Cohomology of classifying spaces of projective unitary groups and related groups

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The projective unitary group $PU(n)$ is the quotient group of the unitary group $U(n)$ by its center S^1, the subset of scalar matrices. Although the cohomology of $PU(n)$ is known by Baum and Browder, the computation of the cohomology of its classifying space remains an open problem. In this talk, I will attempt to explain the importance of the cohomology of the classifying space of the projective unitary group $PU(n)$ from multiple perspectives.

(1) In algebraic topology, the computation of the cohomology of the classifying spaces of simply-connected simple Lie groups has been a classical problem. If the integral homology of the Lie group G has no torsion, Borel gave a satisfactory answer. Borel showed that the cohomology is detected by the maximal torus, and it could be described in terms of rings of invariants. However, if the integral homology of G has torsion, the problem remains open. I will explain the current status of this problem and discuss why I believe the cohomology of the classifying space of certain groups related to the projective unitary groups may be useful.

(2) In algebraic geometry, the torsion of the third integral cohomology group of a smooth complex variety is known as the Brauer group. It is a birational invariant that plays an important role in algebraic geometry. The third integral cohomology class of a topological space is represented by a map to the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$. We discuss the lifting problem and related computations associated with the fibration $BPU(n) \to K(\mathbb{Z}, 3)$. These have been studied in conjunction with the period-index problem by several authors.

(3) Algebraic geometry studies smooth complex projective varieties. In general, a classifying space cannot be a smooth complex projective variety. However, for a finite group G or a connected compact Lie group G and for any positive integer k, there is a smooth complex projective variety X with a k-equivalence map $f\colon X \to BG \times BS^1$. Applying this construction for elementary abelian groups, Atiyah and Hirzebruch constructed counterexamples for the integral Hodge conjecture. I will explain that the special orthogonal group $SO(4)$ gives counterexamples for the integral Hodge conjecture modulo torsion and that the projective unitary group $PU(4)$ provides a new smooth complex projective variety that shows nontriviality of certain new birational invariant associated with coniveau filtrations.