On the classification of quasitoric manifolds

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Preliminaries

Hereafter, T^n always denotes the compact torus $(S^1)^n$.

- T^n always acts on $\mathbb{C}^n$ by
$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) := (t_1 z_1, \dots, t_n z_n).$$
This is called the **standard action of T^n**.
- $T^n \curvearrowright X, Y$.
$f \colon X \to Y$ is **weakly equivariant** $\overset{\text{def}}{\iff}$
$$\exists \psi \in \mathrm{Aut}\,(T^n),\ \forall t \in T^n,\ \forall x \in X,\ f(t \cdot x) = \psi(t) \cdot f(x).$$
- A **(convex) polytope** is $P = \mathrm{Conv}\,(p_1, \dots, p_\ell)$ for finite points $p_1, \dots, p_\ell$ in $\mathbb{R}^m$, e.g. Δ^n, I^n, convex polygons.
$\dim P$ is defined as the dimension of the affine hull of P.
- A polytope is called **simple** if it can naturally be regarded as manifold with corners, i.e. locally modeled in $(\mathbb{R}_{\geq 0})^n$.
The above examples are simple.

Definition of quasitoric manifold

A quasitoric manifold M is defined as follows.

- M : $2n$–dim (closed) smooth manifold
- $T^n \curvearrowright M$: smooth
- P : n–dim simple polytope (regarded as a mfd with corners)

Definition (Davis–Januszkiewicz 1991)

M is a **quasitoric manifold over** P if

(i) $[T^n \curvearrowright M] \overset{\text{local}}{\cong} [T^n \curvearrowright \mathbb{C}^n]$: weakly equivariant diffeo,

(ii) $M/T^n \cong P$: homeo as manifolds with corners.

Actually, the condition (i) is more common than it looks.

Fact (Masuda–Panov 2006)

The condition (i) holds if $T^n \curvearrowright M$ has a fixed point and $H^{\mathrm{odd}}(M) = 0$.

Example: $\mathbb{C}P^2$

For example, $\mathbb{C}P^2$ is a quasitoric manifold over Δ^2 with $T^2 \curvearrowright \mathbb{C}P^2$ by $(t_1, t_2) \cdot [z_0 : z_1 : z_2] := [z_0 : t_1 z_1 : t_2 z_2]$.

Condition (i)

$[T^2 \curvearrowright \mathbb{C}P^2] \overset{\text{local}}{\cong} [T^2 \curvearrowright \mathbb{C}^2]$: weakly equivariant diffeo.

Suppose that

- $t = (t_1, t_2) \in T^2$,
- $z = [z_0 : z_1 : z_2] \in \mathbb{C}P^2$,
- $U_i = (z_i \neq 0)$, $\varphi_i \colon U_i \to \mathbb{C}^2$: the standard chart.

First consider $\varphi_0 \colon U_0 \to \mathbb{C}^2$.

Since $U_0 = \{[1 : z_1 : z_2]\}$,

$$\varphi_0(t \cdot z) = \varphi_0([1 : t_1 z_1 : t_2 z_2]) = (t_1 z_1, t_2 z_2) = t \cdot \varphi_0(z).$$

For $\varphi_1\colon U_1 \cong \mathbb{C}^2$, since $U_1 = \{[z_0 : 1 : z_2]\}$,

$$\begin{aligned}\varphi_1(t \cdot z) &= \varphi_1([z_0 : t_1 : t_2 z_2]) &&= \varphi_1([t_1^{-1} z_0 : 1 : t_1^{-1} t_2 z_2]) \\ &= (t_1^{-1} z_0, t_1^{-1} t_2 z_2) &&= \psi_1(t) \cdot \varphi_1(z)\end{aligned}$$

where $\psi_1(t_1, t_2) = (t_1^{-1}, t_1^{-1} t_2)$, an automorphism of T^2.

Similarly,

$$\varphi_2(t \cdot z) = \psi_2(t) \cdot \varphi_2(z)$$

where $\psi_2(t_1, t_2) = (t_2^{-1}, t_1 t_2^{-1})$, an automorphism of T^2.

Remark

$\mathrm{Aut}\,(T^n) = \mathrm{Aut}\,(\mathbb{R}^n/\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$.

Through this identification, $\psi_1 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

Condition (ii)

$\mathbb{C}P^2/T^2 \cong \Delta^2$: homeo as manifolds with corners.

$\mathbb{C}P^2/T^2 = S^5/T^3$.
Then the moment map $(z_0, z_1, z_2) \mapsto (|z_0|, |z_1|, |z_2|)$ descends to a homeo $S^5/T^3 \to S^2 \cap (\mathbb{R}_{\geq 0})^3 \cong \Delta^2$.

Remark

Similarly, $\mathbb{C}P^n$ is a quasitoric manifold over Δ^n.

Fact

More generally, any projective non-singular toric variety is a quasitoric manifold.

Fundamental theorem

Theorem (Davis–Januszkiewicz 1991)

There is the following one-to-one correspondence:

Geometry (Topology)		Combinatorics
{a quasitoric manifold}	$\longleftrightarrow$	{a characteristic pair}
({a toric variety}	$\longleftrightarrow$	{a fan})

A **characteristic pair** is (P, λ) where

(i) P is an n–dim simple polytope with m facets $F_1, \dots, F_m$,

(ii) $\lambda = (\lambda_1, \dots, \lambda_m)$ is an $(n \times m)$–matrix of integers

satisfying certain conditions, where a facet means a codim 1 face.

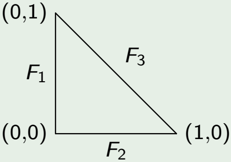
(e.g. Δ^n has $n+1$ facets, each of which is isomorphic to Δ^{n-1}.)

This λ is called a **characteristic matrix on** P.

Example: For $\mathbb{C}P^2$

(P, λ) corresponding to $\mathbb{C}P^2$ is obtained as follows.

(i) $P = \mathbb{C}P^2/T^2 = \Delta^2$, whose facets are:



If we denote the projection by $\pi\colon \mathbb{C}P^2 \to \Delta^2$,

$$\pi^{-1}(F_1) = \{[z_0 : 0 : z_2]\}, \pi^{-1}(F_2) = \{[z_0 : z_1 : 0]\},$$
$$\pi^{-1}(F_3) = \{[0 : z_1 : z_2]\}.$$

(ii) The isotropy subgroups of $T^2 \curvearrowright \pi^{-1}(F_i)$ are $\{(t, 1)\}$, $\{(1, t)\}$, $\{(t, t)\}$ respectively.

$$\lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since $T^n \curvearrowright M$ is locally standard, the characteristic pair (P, λ) contains all the information of isotropy subgroups.

$$
\begin{tikzcd}
M \arrow[r, equal, "\text{locally}", "\text{(through } \psi)"'] \arrow[d] & \mathbb{C}^n \arrow[d] \\
P \arrow[r, equal, "\text{locally}"] & (\mathbb{R}_{\geq 0})^n
\end{tikzcd}
$$

$$\text{faces} \longleftrightarrow \text{strata}$$

Moreover, through the blowup $M \to \widetilde{M} \cong T^n \times P$, we obtain a section $P \to M$ and therefore $M \cong (T^n \times P)/\sim_{\text{isotropy}}$.

Thus we see that the characteristic pair (P, λ) determines M.

Conversely, we can construct a quasitoric manifold $M(P, \lambda)$ by putting $M(P, \lambda) := (T^n \times P)/\sim_\lambda$.
Hereafter, we simply denote $M(P, \lambda)$ by $M(\lambda)$.

Page for clarification

First, the precise definition of characteristic matrix on P is an integer $(n \times m)$–matrix $\lambda = (\lambda_1, \dots, \lambda_m)$ with

$$|\lambda_{i_1}, \dots, \lambda_{i_n}| = \pm 1 \quad \text{if} \quad F_{i_1} \cap \cdots \cap F_{i_n} = \text{a vertex}.$$

Second, to state the fundamental theorem of quasitoric manifolds more precisely, there is a bijection

$$\frac{\{\text{characteristic matrix on } P\}}{\sim} \to \frac{\{\text{quasitoric manifold over } P\}}{\text{weakly equivariant homeo}}.$$

Roughly, $\sim$ is defined by left multiplication of $\mathrm{GL}_n(\mathbb{Z})$ and right action of signed permutations associated with P.

Anyway, through the correspondence, the classification of quasitoric manifolds up to weakly equivariant homeo reduces to elementary (but not necessarily easy) calculations.

Another construction

A quasitoric manifold M over P can be constructed as a quotient space of the **moment-angle manifold** $\mathcal{Z}_P$.

Let K be a simplicial complex on $[m] = \{1, \ldots, m\}$ and (X, A) a topological pair. For $\sigma \in K$, we put

$$Z_\sigma(X, A) := \{(z_1, \ldots, z_m) \in X^m \mid z_i \in A \text{ if } i \notin \sigma\}.$$

Then $Z_K(X, A) := \bigcup_{\sigma \in K} Z_\sigma(X, A)$ is called a **polyhedral product**.

Definition (moment-angle complex)

$\mathcal{Z}_K := Z_K(D^2, S^1)$ is called a **moment-angle complex**, and $DJ_K := Z_K(\mathbb{C}P^\infty, *)$ is called a **Davis–Januszkiewicz space**.

For an n–dim simple polytope P with m facets $F_1, \ldots, F_m$,

$$K_P := \{\, \{i_1, \ldots, i_k\} \subset [m] \mid F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset \,\}$$

and $\mathcal{Z}_P := \mathcal{Z}_{K_P}$, which turns out to be a smooth manifold.

From a characteristic matrix λ on P, we can construct a quasitoric manifold $M(\lambda)$ as follows.

(1) Since λ is an integer matrix, the linear map $\lambda\colon \mathbb{R}^m \to \mathbb{R}^n$ descends to a homomorphism $\bar{\lambda}\colon \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^n/\mathbb{Z}^n$.

(2) Through the identification $T^k = \mathbb{R}^k/\mathbb{Z}^k$, we obtain a subtorus $T_\lambda := \ker\bar{\lambda} \subset T^m$, which is isomorphic to T^{m-n}.

(3) The action $T^m \curvearrowright (D^2)^m$ restricts to a free action $T_\lambda \curvearrowright \mathcal{Z}_P$, and $M(\lambda) := \mathcal{Z}_P/T_\lambda$ is a quasitoric manifold over P.

Remark

The one-to-one correspondence in the fundamental theorem is also given by this construction $\lambda \mapsto M(\lambda)$.

For example, $\lambda = (-E_n\ \mathbf{1})$ is a characteristic matrix on Δ^n, and $T_\lambda \subseteq T^{n+1}$ is the diagonal 1-dim torus, $\mathcal{Z}_{\Delta^n} = \partial(D^2)^{n+1} \cong S^{2n+1}$. Thus this construction recovers $\mathbb{C}P^n = S^{2n+1}/T_\lambda$.

Cohomology

In conclusion, the (co)homology of a quasitoric manifold is computed as follows.

Theorem (Davis–Januszkiewicz 1991)

M : a quasitoric manifold
(P, λ) : the characteristic pair corresponding to M

(1) $H_{\mathrm{odd}}(M) = 0$, $H_{\mathrm{even}}(M)$ is free and $b_{2i}(M) = h_i(P)$.
In particular, $b_2(M) = m - n$.

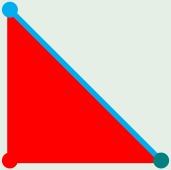
(2) $H^*(M) \cong \mathbb{Z}[v_1, \ldots, v_m]/(\mathcal{I}_P + \mathcal{J}_\lambda)$ where $\deg v_i = 2$ and

$$\mathcal{I}_P = (v_{i_1} \cdots v_{i_k} \mid F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset),$$
$$\mathcal{J}_\lambda = (\lambda_{i,1} v_1 + \cdots + \lambda_{i,m} v_m \mid i = 1, \ldots, n).$$

As for (1), a quasitoric manifold M has a cell decomposition with only even-dimensional cells.

Example: For $\mathbb{C}P^2$

Let $\pi\colon \mathbb{C}P^2 \to \Delta^2 \cong \mathbb{C}P^2/T^2$ be the projection.
Then we can recover the usual decomposition $\mathbb{C}P^2 = e^0 \cup e^2 \cup e^4$ by $e^0 := \pi^{-1}(■)$, $e^2 := \pi^{-1}(■)$, $e^4 := \pi^{-1}(■)$.



$H_*(M)$ can be computed simply by this kind of decomposition.

As for (2), $H^*(M)$ can be computed by the Leray–Serre spectral sequence for the following fiber sequence.

$$M \to ET^n \times_{T^n} M \to BT^n$$

Since $M \cong \mathcal{Z}_P/T_\lambda$, we have

$$\begin{aligned} ET^n \times_{T^n} M \;\cong\; ET^m \times_{T^m} \mathcal{Z}_P &= \bigcup_{\sigma \in K_P} ET^m \times_{T^m} Z_\sigma(D^2, S^1) \\ &\simeq \bigcup_{\sigma \in K_P} Z_\sigma(\mathbb{C}P^\infty, *) \;=\; DJ_{K_P}. \end{aligned}$$

$H^*(DJ_{K_P}) \cong \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}_P$, and the ideal generate by $H^+(BT^n) \to H^+(DJ_{K_P})$ coincides with $\mathcal{J}_\lambda$.

Theorem

(2) $H^*(M) \cong \mathbb{Z}[v_1, \dots, v_m]/(\mathcal{I}_P + \mathcal{J}_\lambda)$ where $\deg v_i = 2$ and

$$\mathcal{I}_P = (v_{i_1} \cdots v_{i_k} \mid F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset),$$
$$\mathcal{J}_\lambda = (\lambda_{i,1} v_1 + \cdots + \lambda_{i,m} v_m \mid i = 1, \dots, n).$$

§2. Cohomological rigidity problem

- Statement
- Supporting facts

Statement

M, M': quasitoric manifolds

Theorem (Masuda 2008)

M and M' are equivariantly homeo
$\iff H^*_T(M) \cong H^*_T(M')$ as $H^*(BT)$–algebras.

Problem (Masuda 2008, Masuda–Suh 2006 for toric manifolds)

Are M and M' homeo if $H^*(M) \cong H^*(M')$ as graded rings?

There are many partial classification results on quasitoric manifolds (which we will see later) and this rigidity holds for all of them.
Conversely, we haven't found any counterexample to this problem until now.

Supporting facts

First, the rational homotopy version is solved affirmatively:

Theorem (Panov–Ray 2008)

A quasitoric manifold is rationally formal.

Second, since the cohomology rings are generated by degree 2 elements, any isomorphism preserves the Steenrod operations.

Similarly, any cohomology isomorphism lifts to K-theory isomorphism along Chern character, and therefore cohomology isomorphism includes the information of Adams e-invariant.

$$\begin{tikzcd} K(M) \arrow[r, "\tilde{\varphi}"] \arrow[d, "\mathrm{ch}"'] & K(M') \arrow[d, "\mathrm{ch}"] \\ H^*(M;\mathbb{Q}) \arrow[r, "\varphi\otimes\mathbb{Q}"] & H^*(M';\mathbb{Q}) \end{tikzcd}$$

§3. Classification results

- I. Over specified P
- II. Bundle-type
- III. Vertex cut

I. Over specified P

P : n–dim simple polytope with m facets

In the following cases, the classification of quasitoric manifolds over P had been done.

- $n = 1$ ($\iff \dim M = 2$) : $P = I$, the interval.
- $n = 2$ ($\iff \dim M = 4$) : P is a convex m–gon.
- $m - n = 1$ ($\iff b_2(M) = 1$) : $P = \Delta^n$.
- $m - n = 2$ ($\iff b_2(M) = 2$) : $P = \Delta^k \times \Delta^{n-k}$ for some k.
- $n = m - n = 3$ ($\iff \frac{\dim M}{2} = b_2(M) = 3$) : $P = I^3, C^3(6)^*$.

Here $C^n(m)$ denotes the n–dim **cyclic polytope** with m vertices (we will see later) and $C^n(m)^*$ means its dual.

Additionally, the quasitoric manifolds over $C^n(m)^*$ are classified unless $n = 3$ and $m \geq 7$.

I-1. Over Δ^n

Theorem (Davis–Januszkiewicz 1991)

A quasitoric manifold with $b_2 = 1$ is homeo to $\mathbb{C}P^n$.

Let $\lambda = (A\ \mathbf{b})$ be a characteristic matrix on Δ^n.
Then we see $M(\lambda) \cong \mathbb{C}P^n$ by the fundamental theorem and the following computation.

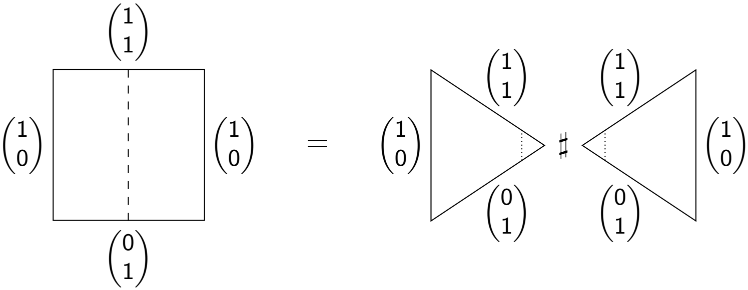
$$\lambda = A(E_n\ A^{-1}\mathbf{b}) \sim (E_n\ A^{-1}\mathbf{b}) = \begin{pmatrix} 1 & & O & \pm 1 \\ & \ddots & & \vdots \\ O & & 1 & \pm 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \pm 1 & & O & 1 \\ & \ddots & & \vdots \\ O & & \pm 1 & 1 \end{pmatrix} \sim (E_n\ \mathbf{1})$$

Theorem (Orlik–Raymond 1970, H 2015)

Any 4–dim quasitoric manifold is homeo to a connected sum of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$.

This theorem was first proven as an immediate consequence of the classical theorem of Orlik and Raymond, but we can also prove it by using **connected sum of characteristic pairs**.



I-3. Over $\Delta^k \times \Delta^{n-k}$

Theorem (Choi–Park–Suh 2012)

Quasitoric manifolds M, M' with $b_2 = 2$ are homeo if $H^*(M) \cong H^*(M')$.

For each dimension, these manifolds are $\mathbb{C}P^k$–bundles over $\mathbb{C}P^{n-k}$ except finite cases, and therefore computation of cohomology is comparatively easy.
We can check that cohomology isomorphism implies equivalence of characteristic matrices in most cases.
Then we only have to construct homeomorphisms (not weakly equivariant) in finite exceptional cases, and this is done by ad hoc construction via $\mathcal{Z}_{\Delta^k \times \Delta^{n-k}} = S^{2k+1} \times S^{2n-2k+1}$.

Theorem (H 2017)

6–dim quasitoric manifolds M, M' with $b_2 = 3$ are homeo if $H^*(M) \cong H^*(M')$.

This theorem is due to careful computation of cohomology rings and the classification theorem of 6–dim manifolds.

Moreover, for 6–dim cases, the following is also known.

Theorem (Buchstaber–Erokhovets–Masuda–Panov–Park 2017)

Let M, M' be 6–dim quasitoric manifolds over P, P' which are flag and have no 4–belt. Then M and M' are weakly equivariantly diffeo if $H^*(M) \cong H^*(M')$.

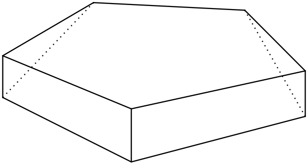
Here a 4–belt is a sequence $F_1, \ldots, F_4$ of facets (indexed by $\mathbb{Z}/4$) s.t. $F_i \cap F_{i+1} \neq \emptyset$ and $F_i \cap F_{i+2} = \emptyset$ $(i \in \mathbb{Z}/4)$.

Taking $t_1 < t_2 < \cdots < t_m$, the **cyclic polytope** $C^n(m)$ is defined by

$$C^n(m) := \mathrm{Conv}\left(\begin{pmatrix} t_1 \\ t_1^2 \\ \vdots \\ t_1^n \end{pmatrix}, \ldots, \begin{pmatrix} t_m \\ t_m^2 \\ \vdots \\ t_m^n \end{pmatrix} \right).$$

Then the dual $C^n(m)^*$ is a simple polytope. ($C^n(m)$ is not necessarily simple itself.)

For example, $C^3(6)^*$ is:



For the cyclic polytopes of dim ≥ 4, we have the following.

Theorem (H 2015)

Quasitoric manifolds M, M' over the dual cyclic polytope $C^n(m)^*$ ($n \geq 4$) are weakly equivariantly homeo if $H^*(M) \cong H^*(M')$.

In more detail, up to weakly equivariant homeo, there are

- 4 quasitoric manifolds over $C^4(7)^*$,
- 46 quasitoric manifolds over $C^5(8)^*$, and
- no quasitoric manifold over the other $C^n(m)^*$ with $n \geq 4$ and $m \geq n + 3$.

Then the rest is the cases $n = 3$ and $m \geq 7$, which are still open.
(The other cases are included in the previous results.)

As in the case of $\Delta^k \times \Delta^{n-k}$, quasitoric manifolds over product polytope often have fiber bundle structure.
The most prominent examples are the following:

Definition

An n–**stage Bott manifold** B_n is an iterated $\mathbb{C}P^1$–bundle

$$B_n \xrightarrow{p_{n-1}} B_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_2} B_2 \xrightarrow{p_1} B_1 \xrightarrow{p_0} B_0 = *$$

where each p_i is $P(L_i \oplus \mathbb{C}) \to B_i$ for some line bundle L_i.
An n–stage Bott manifold is a quasitoric manifold over I^n.

example

A 2–stage Bott manifold is a Hirzebruch surface, which is diffeo to $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$.

The cohomological rigidity of Bott manifolds had been studied progressively, and now we have the following conclusion.

Theorem (Choi–Hwang–Jang 2025)

Any cohomology ring isomorphism between Bott manifolds is induced from a diffeomorphism.

On the other hand, there are some classification results on other bundle-type quasitoric manifolds, for example:

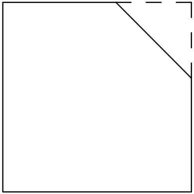
Theorem (H 2017)

Any cohomology ring isomorphism between $(\mathbb{C}P^2\sharp\mathbb{C}P^2)$–bundle type quasitoric manifolds is induced from a weakly equivariant homeomorphism.

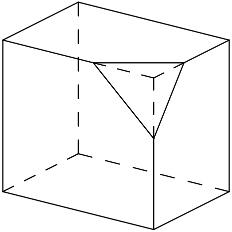
Note that $\mathbb{C}P^2\sharp\mathbb{C}P^2$ is the only quasitoric manifold over I^2 without bundle structure, and $(\mathbb{C}P^2\sharp\mathbb{C}P^2)$–bundle type quasitoric manifolds are quasitoric manifolds over I^{2n}.

III. Vertex cut

Let $\mathrm{vc}(I^n)$ be the polytope obtained by cutting off a vertex from I^n:



$\mathrm{vc}(I^2)$

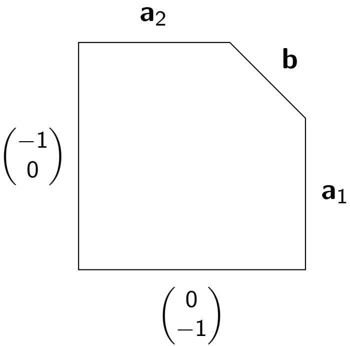


$\mathrm{vc}(I^3)$

The toric manifolds over $\mathrm{vc}(I^n)$ are known to be cohomologically rigid. (We will see it in more detail later.)

In the proof, we use an extension of connected sum method. (Note that $\mathrm{vc}(I^n) = I^n \sharp \Delta^n$.)

A toric manifold over $\mathrm{vc}(I^n)$ corresponds to a characteristic matrix of the form $(-E_n, A, \mathbf{b})$ with some additional conditions.



If $\det A = \pm 1$, this toric manifold decomposes into the connected sum $M(-E_n, A) \sharp M(A, \mathbf{b}) \cong M(-E_n, A) \sharp \mathbb{C}P^n$.

As mentioned in the previous page, we can extend this kind of argument to the cases $\det A = 0, 2$.

Put

$$\mathcal{VAR}^n := \{\text{toric manifold over } \mathrm{vc}(I^n)\}/\text{variety isom}$$
$$\mathcal{VAR}^n(q) := \{X \in \mathcal{VAR}^n \mid \det X = q\}$$

where $\det X := \det A$.

Theorem (H–Kuwata–Masuda–Park 2018)

For $X, Y \in \mathcal{VAR}^n$, $H^*(X) \cong H^*(Y)$ only if $\det X = \det Y$. Moreover,

(1) if $q \neq 0, 1, 2$, $\mathcal{VAR}^n(q)$ consists of one element;

(2) if $q = 0, 2$, $\mathcal{VAR}^n(q)$ consists of finite elements diffeomorphic to each other;

(3) $\mathcal{VAR}^n(1)$ consists of the blow-ups of Bott manifolds.

In particular, the classification of toric manifolds over $\mathrm{vc}(I^n)$ reduces to that of Bott manifolds, which is now solved by Choi, Hwang, and Jang as mentioned before.

§4. Other approaches

- On stable homotopy types
- Homotopy commutativity

On stable homotopy types

Let M be a quasitoric manifold over P, and take $u \in \mathbb{Z}$.
Recall that $M \cong \mathcal{Z}_P / T_\lambda$, where $\mathcal{Z}_P \subseteq (D^2)^m$.
Then $(z_1, \ldots, z_m) \mapsto ({z_1}^u, \ldots, {z_m}^u)$ defines a map $\mathcal{Z}_P \to \mathcal{Z}_P$, and it descends to a map $M \to M$.
By using this power map, we obtain the following decomposition.

Corollary (H–Kishimoto–Sato 2016)

$\Sigma M_{(p)} \simeq \bigvee_{i=1}^n \left(\bigvee^{b_{2i}} S^{2i+1}_{(p)} \right)$ for any prime $p > \dim P$.

On the other hand, for two quasitoric manifolds M, N, we obtain the following by using Adams e-invariant.

Theorem (H–Kishimoto 2017)

If $\dim M = 2n$, $H^*(M; \mathbb{Z}) \cong H^*(N; \mathbb{Z})$ and $p^2 - 2 \geq n$, then $M_{(p)}$ and $N_{(p)}$ are stably homotopy equivalent.

Homotopy commutativity

Let M be a quasitoric manifold over P, whose the ch matrix is λ. We determined the necessary and sufficient condition for ΩM to be homotopy commutative in terms of P and λ.

Theorem (H–Kishimoto–Tong–Tsutaya 2026)

ΩM is homotopy commutative if and only if $P \cong (\Delta^3)^k$ and

$$\lambda \simeq \begin{pmatrix} E_3 & a_{11} & & a_{12} & & a_{13} & & & a_{1k} \\ & a_{21} & E_3 & a_{22} & & a_{23} & & & a_{2k} \\ & a_{31} & & a_{32} & E_3 & a_{33} & & & a_{3k} \\ & \vdots & & \vdots & & \vdots & \ddots & & \vdots \\ & a_{k1} & & a_{k2} & & a_{k3} & & E_3 & a_{kk} \end{pmatrix} \qquad (1)$$

for the identity matrix E_3 and $a_{ij} \in \mathbb{Z}^3$ such that

$$a_{ii} = {}^t(1,1,1) \quad \text{and} \quad (1,1,1)a_{ij} \equiv 0 \mod 2 \quad (i \neq j). \qquad (2)$$

To explain the proof very roughly, since there is a fibration $T_\lambda \to \mathcal{Z}_P \to M$, ΩM decomposes into $\Omega\mathcal{Z}_P \times T_\lambda$.

Then ΩM is homotopy commutative only if so is $\Omega\mathcal{Z}_P$, and the latter can be translated into combinatorial properties of P.

Thus we see that P must be $(\Delta^3)^k$, and computing Samelson products in $\Omega\mathcal{Z}_P \times T_\lambda = (\Omega S^7)^k \times (S^1)^k$, we obtain the rest part.

(This argument finally comes down to computation of Steenrod squares in $H^*(M;\mathbb{Z}/2)$. Namely, the part $(1,1,1)a_{ij} \equiv 0 \bmod 2$ means vanishing of some Steenrod squares.)