Conditionally convergent spectral sequences

[Boa] Boardman, Conditionally convergent spectral sequences, 1999

[DK] Davis-Kirk, Lecture notes in algebraic topology

[Wei] Weibel, An introduction to homological algebra

[FHT] Félix-Halperin-Thomas,
Rational homotopy theory, GTM 205

[NoRef] 特に参考文献がなく、このnoteの著者(若月)が自分で考えた内容であることを示す.
ただし、どれもプロにはwell-knownだと思う.
(なので[W.]とは書かなかった)

Contents

Aim [Boa]をちゃんと理解する.

- 主に§6,7の"half-plane spectral sequences"が目標
- [Boa]との差分:
  - gapを埋めたり、証明を整理したりした
  - filtered cpxによる構成とexact coupleによる構成の比較を追加(§2.3)
  - exampleを追加(とくに§1.2, §5.1)
  - [Boa]の内容のうち、以下は扱わない:
    §4. Homotopy limits and colimits of spectra
    §11. Multicomplexes
    §13. Serre s.s. of a fibration
    §14. Bockstein s.s.
    §15. Adams s.s.

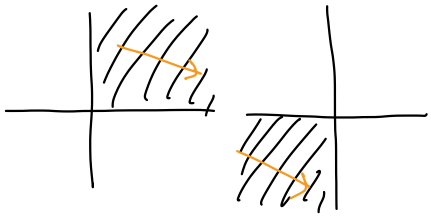
Notation

- $\mathbb{K}$: comm. ring $(\neq 0)$
- M: graded $\mathbb{K}$-mod $\underset{\text{def}}{\Leftrightarrow}$ $M = \{M^n\}_{n\in\mathbb{Z}}$
  (NOT $M = \bigoplus_n M^n$)
  $x \in M \underset{\text{def}}{\Leftrightarrow} \exists n.\ x \in M^n$
  (we only consider homogeneous elements)
- C: complex
  $\underset{\text{def}}{\Leftrightarrow}$ $d : C^n \to C^{n+1}$, $d\circ d = 0$
- (co)limは常にdegreewiseにとる
  $\cdots \to M^{i+1} \to M^i \to \cdots$: seq. of gr. mod's
  $(M^i = \{M^{i,n}\}_{n\in\mathbb{Z}}$: gr. mod$)$
  $\rightsquigarrow$ $\lim_i M^i \underset{\text{def}}{:=} \{\lim_i M^{i,n}\}_{n\in\mathbb{Z}}$
- このnoteでは、"spectral sequence"は常に
  「unrolled exact coupleから
  §2.2の方法で構成したもの」
  を表すことにする.
  (ただし§1, §2.3だけは例外)

§0. Introduction

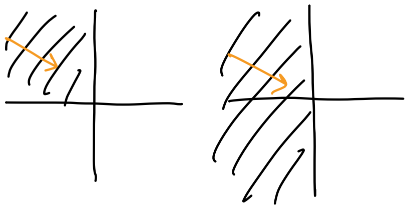
① Types of spectral sequences

- bounded (§1.1)



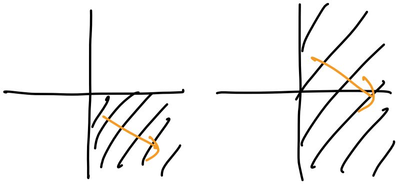
⇒ 自明に収束

- half-plane with exiting differentials (§4.2)



⇒ OK
（少し非自明）

- half-plane with entering differentials (§4.3)



⇒ ヤバい
（収束しないこともある
see §1.2 for example）

Thm 4.3.1 (rough ver.)

Consider
$\{E_r^{st}\}$: half-plane s.s. with entering diff.

Assume
(a) $\{E_r^{st}\}$ is "conditionally convergent"
(b) $\forall s,t$, $RE_\infty^{st}=0$

Then
$\{E_r^{st}\}$ is "strongly convergent"

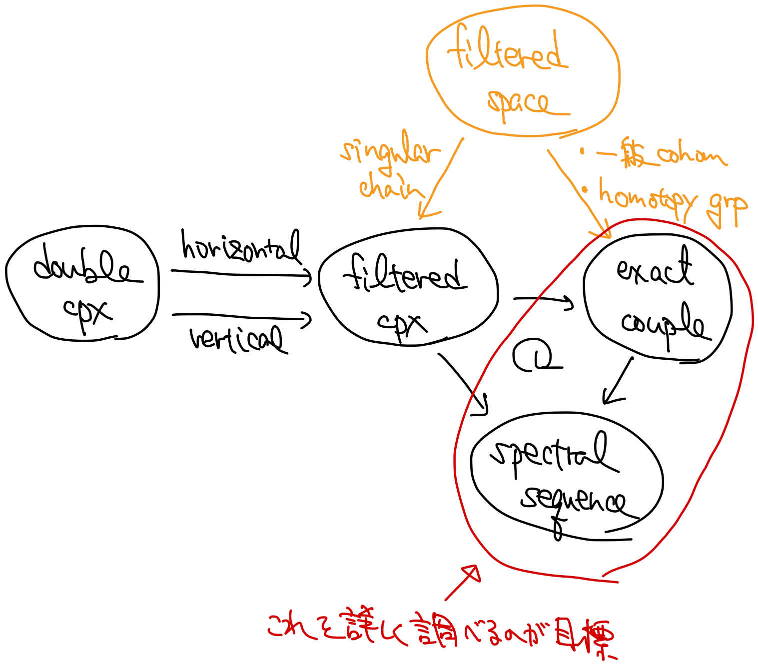
← (a) structural condition
- s.s. をつくる人が check する
- $\{E_r\}$ の情報だけでは分からない

(b) internal condition
- s.s. の user が check する
- $\{E_r\}$ の情報だけで分かる

Rmk

上では色々な部分をあいまいにしている
- exiting / entering diff の定義 ← ~~E_2 が上図~~の形
- 各種 convergece の定義
  （とくに収束先について）

② Constructions of spectral sequences



§1 Spectral sequences for filtered complexes

この section の内容は [Boa] には書かれていない

(☹ too easy)

§1.1 Review on spectral sequences for filtered complexes

Reference

- 服部，位相幾何学
- 河田，ホモロジー代数
- McCleary, User's guide to s.s.

etc…

ただし，収束まわりの仮定を弱めたりしている

Def 1.1.1

$\{F^sC\}_{s\in\mathbb{Z}}$: filtered complex

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ C : \text{cpx} \\ \cdot\ \cdots \subset F^{s+1}C \subset F^sC \subset \cdots \subset C \end{cases}$$

sequence of subcpxs

We fix

$\{F^sC\}$: filtered cpx

Def 1.1.2

- $F^{\infty}C := \lim_s F^sC = \bigcap_s F^sC$

  $F^{-\infty}C := \operatorname{colim}_s F^sC = \bigcup_s F^sC$

- For $-1 \leq r \leq \infty$, define

  $$\widetilde{Z}_r^{st} := F^sC^{s+t} \cap d^{-1}(F^{s+r}C^{s+t+1})$$

  $$\widetilde{B}_r^{st} := F^sC^{s+t} \cap d(F^{s-r}C^{s+t-1})$$

- For $0 \leq r \leq \infty$, define

  $$\widetilde{E}_r^{st} := \frac{\widetilde{Z}_r^{st}}{\widetilde{Z}_{r-1}^{s+1,t-1} + \widetilde{B}_{r-1}^{st}}$$

- For $0 \leq r < \infty$, define

  $$\begin{array}{cccc} d_r^{st} : & \widetilde{E}_r^{st} & \longrightarrow & \widetilde{E}_r^{s+r,t-r+1} \\ & [x] & \longmapsto & [dx] \end{array}$$

  (well-def'd)

  $(x \in \widetilde{Z}_r^{st})$

$$\widetilde{B}_{-1}^{st} \subset \widetilde{B}_0^{st} \subset \widetilde{B}_1^{st} \subset \cdots \subset \widetilde{B}_\infty^{st} \subset F^sC^{s+t} \cap \operatorname{Im} d$$

$$\subset F^sC^{s+t} \cap \operatorname{Ker} d \subset \widetilde{Z}_\infty^{st} \subset \cdots \subset \widetilde{Z}_1^{st} \subset \widetilde{Z}_0^{st} \subset \widetilde{Z}_{-1}^{st}$$

Rmk 1.1.3

§2 で exact couple に対して $Z_r^{st}, B_r^{st}, E_r^{st}$ を def する。 $(1 \leq r \leq \infty)$

tilde をつけたのは，これらと区別するため。

- $\widetilde{Z}_r^{st}$ と Z_r^{st} / $\widetilde{B}_r^{st}$ と B_r^{st} は全くの別物
- $1 \leq r < \infty$ なら $\widetilde{E}_r^{st} \simeq E_r^{st}$ (see Cor 2.3.3)
- $r = \infty$ だと $\widetilde{E}_r^{st} \neq E_r^{st}$ (see Rmk 2.3.4)

Prop 1.1.4

(1) $H^{st}(\widetilde{E}_r, d_r) \cong \widetilde{E}_{r+1}^{st}$ $\quad (0 \overset{\forall}{\leq} r < \infty)$

(2) $\widetilde{E}_0^{st} = F^sC^{s+t} / F^{s+1}C^{s+t}$

(3) $\widetilde{E}_1^{st} = H^{s+t}(F^sC / F^{s+1}C)$

$$\begin{array}{ccc} d_1^{st} : H^{s+t}(F^sC / F^{s+1}C) & \longrightarrow & H^{s+t+1}(F^{s+1}C / F^{s+2}C) \\ [[x]] & \longmapsto & [[dx]] \end{array}$$

$\left(x \in F^sC^{s+t},\ [x] \in F^sC^{s+t} / F^{s+1}C^{s+t}\right)$

$\left(\underline{\text{i.e.}}\ d_1^s \text{ is connecting hom for } 0 \to F^{s+1}C / F^{s+2}C \to F^sC / F^{s+2}C \to F^sC / F^{s+1}C \to 0\right)$

proof

def から直接計算するだけ。

//

(simplest case で) 収束を議論するために次を def:

Def 1.1.5

- Consider two conditions:

$$\begin{cases} (\mathrm{I}^{st}_r) & F^{s-r+1}C^{s+t-1} = C^{s+t-1} \\ (\mathrm{II}^{st}_r) & F^{s+r}C^{s+t+1} = 0 \end{cases}$$

- $\{F^sC\}$: <u>bounded</u>

$\overset{}{\underset{\mathrm{def}}{\Longleftrightarrow}}$ $\forall s,t$, $\exists r$ s.t. (I^{st}_r), (II^{st}_r)

↑ 多くの文献では bounded を最初から仮定している

Prop 1.1.6

(1) Assume (I^{st}_r). Then

$$\tilde{B}^{st}_{r-1} = \tilde{B}^{st}_r = \cdots = \tilde{B}^{st}_\infty = F^sC^{s+t} \cap \mathrm{Im}\, d$$

(2) Assume (II^{st}_r). Then

$$F^sC^{s+t} \cap \mathrm{Ker}\, d = \tilde{Z}^{st}_\infty = \cdots = \tilde{Z}^{st}_{r+1} = \tilde{Z}^{st}_r$$

$$F^{s+1}C^{s+t} \cap \mathrm{Ker}\, d = \tilde{Z}^{s+1,t-1}_\infty = \cdots = \tilde{Z}^{s+1,t-1}_r = \tilde{Z}^{s+1,t-1}_{r-1}$$

(3) Assume (I^{st}_r) and (II^{st}_r). Then

$$\tilde{E}^{st}_r = \tilde{E}^{st}_{r+1} = \cdots = \tilde{E}^{st}_\infty$$

<u>proof</u> def から直ちに従う. //

$H^*(C)$ と E_∞ を比較したい.

Def 1.1.7

$$F^sH^n(C) := \mathrm{Im}\left(H^n(F^sC) \to H^n(C)\right)$$
$$\subset H^n(C)$$

$\rightsquigarrow$ $H^n(F^sC) \twoheadrightarrow F^sH^n(C)$

Lem 1.1.8

Fix $s,t \in \mathbb{Z}$ and $1 \le r \le \infty$

<u>Then</u> subquotient of C

$$f^{st}_r : H^{s+t}(F^sC) \longrightarrow \tilde{E}^{st}_r \quad : \text{well-def'd}$$
$$[x] \longmapsto [x]$$

<u>proof</u>

- $F^sC^{s+t} \cap \mathrm{Ker}\, d \longrightarrow \tilde{Z}^{st}_r$: well-def'd

(∵ $\overset{\vee}{x}$ をとると.
$x \in F^sC^{s+t} \cap \mathrm{Ker}\, d \subset \tilde{Z}^{st}_\infty \subset \tilde{Z}^{st}_r$)

- $H^{s+t}(F^sC) \longrightarrow \tilde{E}^{st}_r$: well-def'd

(∵ $x = dy$, $y \in F^sC^{s+t-1}$ とする.
$r \ge 1$ より、$y \in F^{s-r+1}C^{s+t-1}$
$\rightsquigarrow$ $x = dy \in \tilde{B}^{st}_{r-1}$
$\rightsquigarrow$ $[x] = [dy] = 0 \in \tilde{E}^{st}_r$)

//

これが下図の maps を induce するのはいつ?

$$\begin{tikzcd}
H^{s+t}(F^sC) \arrow[r, "f^{st}_r"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{E}^{st}_r \\
F^sH^{s+t}(C) \arrow[ur, dashed, "\exists ? g^{st}_r"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \\
F^sH^{s+t}(C) / F^{s+1}H^{s+t}(C) \arrow[uur, dashed, "\exists ? h^{st}_r"'] &
\end{tikzcd}$$

← isom?

Prop 1.1.9

Fix $s, t \in \mathbb{Z}$ and $1 \leq r \leq \infty$

(1) Assume (I_r^{st}). Then f_r^{st} induces

- $g_r^{st} : F^sH^{s+t}(C) \longrightarrow \tilde{E}_r^{st}$
- $h_r^{st} : F^sH^{s+t}(C) / F^{s+1}H^{s+t}(C) \longrightarrow \tilde{E}_r^{st}$

(2) Assume (I_r^{st}) and (II_r^{st}). Then

h_r^{st} : isom.

proof

まず、それぞれを C の subquot として書いておく。

$$\begin{tikzcd}
 & & & \dfrac{\tilde{Z}_r^{st}}{\tilde{Z}_{r-1}^{s+1,t-1}+\tilde{B}_r^{st}} \arrow[d, phantom, "\|"] \\
H^{s+t}(F^sC) \arrow[d] & = & \dfrac{F^sC^{s+t} \cap \mathrm{Ker}\, d}{d(F^sC^{s+t-1})} \arrow[d, two heads] \arrow[r, "f_r^{st}"] & \tilde{E}_r^{st} \\
F^sH^{s+t}(C) \arrow[d] & = & \dfrac{F^sC^{s+t} \cap \mathrm{Ker}\, d}{F^sC^{s+t} \cap \mathrm{Im}\, d} \arrow[d, two heads] \arrow[ur, dashed, "g_r^{st}"] & \\
F^sH^{s+t}(C) / F^{s+1}H^{s+t}(C) & = & \dfrac{F^sC^{s+t} \cap \mathrm{Ker}\, d}{(F^{s+1}C^{s+t} \cap \mathrm{Ker}\, d) + (F^sC^{s+t} \cap \mathrm{Im}\, d)} \arrow[uur, dashed, "h_r^{st}"'] &
\end{tikzcd}$$

(1)

- $F^sC^{s+t} \cap \mathrm{Im}\, d = \tilde{B}_{r-1}^{st} \left(\subset \tilde{Z}_{r-1}^{s+1,t-1} + \hat{B}_{r-1}^{st} \right)$

$$\left(\because \; \tilde{B}_{r-1}^{st} = F^sC^{s+t} \cap d(\underbrace{F^{s-r+1}C^{s+t-1}}_{= \, C^{s+t-1} \; \Leftarrow (\mathrm{I}_r^{st})}) = F^sC^{s+t} \cap \mathrm{Im}\, d \right)$$

- $F^{s+1}C^{s+t} \cap \mathrm{Ker}\, d \subset \tilde{Z}_{r-1}^{s+1,t-1} \left(\subset \tilde{Z}_{r-1}^{s+1,t-1} + \hat{B}_{r-1}^{st} \right)$

$$\left(\because \; \tilde{Z}_{r-1}^{s+1,t-1} = F^{s+1}C^{s+t} \cap \underset{\mathrm{Ker}\, d}{\underset{\cup}{\underbrace{d^{-1}(F^{s+r}C^{s+t+1})}}} \right)$$

↑ これは何の仮定も使ってない

(2) Prop 1.1.6 (1)(2) より、分母分子それぞれが一致する。

$$\left(\begin{array}{l} \text{surj} \Leftarrow (\mathrm{II}_r^{st}) \\ \text{inj} \Leftarrow (\mathrm{I}_r^{st}) \text{ and } (\mathrm{II}_r^{st}) \end{array} \right)$$

← そもそも def に (I_r^{st}) が必要なので、結局両方必要

//

以上で filtered cpx の s.s. の一般論ができた

^ bounded

Thm 1.1.10

$\{F^sC\}$: bounded filtered cpx

Then

we have a s.s. $\{\tilde{E}_r^{st}\}$ s.t.

(1)

$$\begin{tikzcd}
\tilde{E}_1^{st} \arrow[d, "d_1"] \arrow[r, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^{s+t}(F^sC/F^{s+1}C) \arrow[d] \arrow[r, phantom, "\ni"] & {[[x]]} \arrow[d, mapsto] \\
\tilde{E}_1^{s+1,t} \arrow[r, phantom, "\cong"] & H^{s+t+1}(F^{s+1}C/F^{s+2}C) \arrow[r, phantom, "\ni"] & {[[dx]]}
\end{tikzcd}$$

(2) $\forall s, t$, $\exists r$ s.t.

$$\tilde{E}_r^{st} \cong \tilde{E}_{r+1}^{st} \cong \cdots \cong \tilde{E}_\infty^{st}$$

(3) $F^sH^{s+t}(C) / F^{s+1}H^{s+t}(C) \xrightarrow{\cong} \tilde{E}_\infty^{st}$: isom

Rmk 1.1.11

bounded を仮定しない場合は、

$\tilde{E}_\infty^{st}$ については何も分からなくなる

- Thm 1.1.10 (2)(3) は当然不成立.
- とくに (2) について、

$\exists r_0$. $\{E_r\}$: collapse at E_{r_0}

$\left(\text{i.e. } \forall r \geq r_0, \forall s, t, \; d_r^{st} = 0 \right)$

を仮定したとしても $\tilde{E}_{r_0} = \tilde{E}_\infty$ とは限らない

(See §1.2 for examples)

§1.2 Examples of non-convergent spectral sequences

Recall $\mathbb{K}$: comm ring $(\neq 0)$

収束しない s.s. の例を示す。

Def 1.2.1

- $\{D^{st}\}$: double cpx

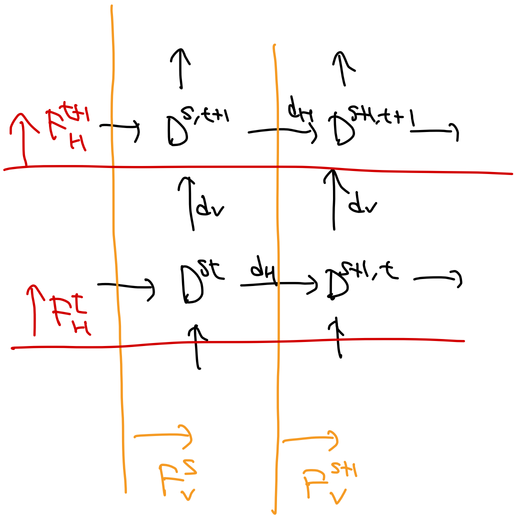
$$\overset{\text{def}}{\Longleftrightarrow}\quad \begin{tikzcd} D^{s,t+1} \arrow[r, "d_H^{s,t+1}"] & D^{s+1,t+1} \\ D^{st} \arrow[u, "d_v^{st}"] \arrow[r, "d_H^{st}"'] \arrow[ur, phantom, "\circlearrowleft(-1)"] & D^{s+1,t} \arrow[u, "d_v^{s+1,t}"'] \end{tikzcd} \quad \text{s.t. } d_H \circ d_v + d_v \circ d_H = 0$$

- $(\mathrm{Tot}\,D, d)$: cpx を次で def:

$$\begin{cases} \cdot\ \mathrm{Tot}^n D = (\mathrm{Tot}\,D)^n := \bigoplus_{s+t=n} D^{st} \\ \cdot\ d := d_v + d_H \end{cases}$$

- $(\mathrm{Tot}\,D, d)$ 上の filtrations $\{F_v^s \mathrm{Tot}\,D\}_s$, $\{F_H^t \mathrm{Tot}\,D\}_t$ を次で def:

$$\begin{cases} \cdot\ F_v^s \mathrm{Tot}^n D := \bigoplus_{\substack{i+j=n \\ i \geq s}} D^{ij} \\ \cdot\ F_H^t \mathrm{Tot}^n D := \bigoplus_{\substack{i+j=n \\ j \geq t}} D^{ij} \end{cases}$$



↪ we have two s.s. $\{\tilde{E}_{v,r}^{st}\}$, $\{\tilde{E}_{H,r}^{st}\}$

$$\begin{cases} d_{vr}^{st} : \tilde{E}_{vr}^{st} \longrightarrow \tilde{E}_{vr}^{s+r,t-r+1} \\ d_{Hr}^{st} : \tilde{E}_{Hr}^{st} \longrightarrow \tilde{E}_{Hr}^{s-r+1,t+r} \end{cases}$$

← 変なnotationだけど仕方ない…？

$$\begin{cases} \tilde{E}_{v,1}^{st} = H^{st}(D, d_v) \\ \tilde{E}_{H,1}^{st} = H^{st}(D, d_H) \end{cases}$$

Ex 1.2.2 [NoRef]

$\{D^{st}\}$: double cpx を次で def:

$$D^{st} := \begin{cases} \mathbb{K} & (s+t = 0, 1 \text{ with } t \leq 0) \\ 0 & (\text{他}) \end{cases}$$

$$d_H^{st} = \begin{cases} \mathrm{id}_{\mathbb{K}} & (s+t = 0 \text{ with } t \leq 0) \\ 0 & (\text{他}) \end{cases}$$

$$d_v^{st} = \begin{cases} -\mathrm{id}_{\mathbb{K}} & (s+t = 0 \text{ with } t < 0) \\ 0 & (\text{他}) \end{cases}$$

$$\left(\begin{tikzcd} D^{00} \arrow[d, phantom, "\shortparallel"] & D^{1,0} \arrow[d, phantom, "\shortparallel"] & & \\ \mathbb{K} \arrow[r, "\mathrm{id}"] & \mathbb{K} & & \\ & \mathbb{K} \arrow[u, "-\mathrm{id}"] \arrow[r, "\mathrm{id}"] \arrow[d, phantom, "\shortparallel"] & \mathbb{K} & \\ & D^{1,-1} & \mathbb{K} \arrow[u, "-\mathrm{id}"] \arrow[r] & {} \\ & & & \ddots \end{tikzcd}\right)$$

$C := \mathrm{Tot}\,D$ と def

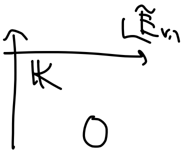
Then we have

- $H^*(C) = 0$ (∵ 直接計算でも良いが、$C = \mathrm{colim}_t F_H^t C$ を使うと楽)
- $\tilde{E}_{H,1}^{st} = 0 \quad (\forall s, t)$
- $\tilde{E}_{v,1}^{st} = \begin{cases} \mathbb{K} & ((s,t) = (0,0)) \\ 0 & (\text{他}) \end{cases}$

Hence:

- $\{\tilde{E}_{H,r}^{st}\}$ については「全て0」なのでちゃんと収束している
- $\{\tilde{E}_{v,r}^{st}\}$ は収束していない:

$$\begin{cases} \cdot\ \text{collapses at } \tilde{E}_{v,1} \cong \mathbb{K} \neq 0 \\ \cdot\ \text{But } H^*(C) = 0 \end{cases}$$



さらに、$\tilde{E}_{v,\infty}$ を計算してみると

$$\tilde{E}_{v,\infty}^{st} = 0 \quad (\forall s, t)$$

$$\left(\begin{array}{l} ∵\ \cdot\ \tilde{Z}_{v,\infty}^{s,-s} = 0 \\ \quad\ \cdot\ \tilde{Z}_{v,\infty}^{s,1-s} = F_v^s C^1 = \bigoplus \mathbb{K} \\ \quad\ \ \ \tilde{B}_{v,\infty}^{s,1-s} = F_v^s C^1 \end{array}\right)$$

↪ $\boxed{\tilde{E}_{v,1}^{00} \neq \tilde{E}_{v,\infty}^{0,0}}$ ← ($\tilde{E}_{v,1}$ で collapse しているにもかかわらず)

Ex 1.2.2 について（後でdefする言葉も使って）まとめると、

- $\{\tilde{E}_{H,r}\}$ は、s.s. with exiting differentials なので収束の問題は何もない。
- $\{\tilde{E}_{V,r}\}$ は、s.s. with entering differentials なので、収束の obstruction がある。

（ $A^\infty = \lim_s H^*(F_V^s C) = \mathbb{K} \neq 0$ なので、$\{\tilde{E}_{V,r}\}$ は conditionally convergent でない ）

← 実は、$C^n = \prod_{p+q=n} D^{pq}$ とすると うまくいく。

最後に、Ex 1.2.2 を一般化しておく。

Ex 1.2.3 [NoRef]

$$\cdots \longrightarrow M^{s+1} \xrightarrow{f^{s+1}} M^s \xrightarrow{f^s} \cdots \longrightarrow M^1 \xrightarrow{f^1} M^0 : \text{seq of } \mathbb{K}\text{-mod's}$$

(Define $M^{-1} := 0$, $f^0 = 0 : M^0 \to 0$)

Define $\{D^{st}\}$ by :

$$\left(\begin{tikzcd}
M^0 \arrow[r, "id"] \arrow[d, phantom, "\shortparallel"] & M^0 \arrow[r, phantom, "="] & D^{1,0} & \\
D^{0,0} & & & \\
& M^1 \arrow[uu, "-f^1"] \arrow[r, "id"] \arrow[d, phantom, "\shortparallel"] & M^1 & \\
& D^{1,-1} & & \\
& & M^2 \arrow[uu, "-f^2"] \arrow[r, "id"] & M^2 \arrow[d, phantom, "\ddots"] \\
& & & {}
\end{tikzcd}\right)$$

$C := \mathrm{Tot}\, D$

Then we have

- $H^*(C) = 0$
- $\tilde{E}_{H,1} = 0$
- $\tilde{E}^{st}_{V,1} = \begin{cases} \mathrm{Ker}\, f^s & (s+t=0,\ s \geq 0) \\ \mathrm{Coker}\, f^s & (s+t=1,\ s \geq 1) \\ 0 & (\text{他}) \end{cases}$

$$\left(\begin{tikzcd}
M^0 \arrow[r, "d_1"] \arrow[rr, bend right=10, "d_2"] \arrow[rrr, bend right=15, "d_3"] & \mathrm{Coker}\, f^1 & & \\
& \mathrm{Ker}\, f^1 & \mathrm{Coker}\, f^2 & \\
& & \mathrm{Ker}\, f^2 & \mathrm{Coker}\, f^3 \\
& & & \ddots
\end{tikzcd}\right)$$

i.e.)
- $\{\tilde{E}_{H,r}\}$ は自明に収束する
- $\{\tilde{E}_{V,r}\}$ は ???

Boardmanの理論を使って、これがちゃんと議論できるようになる。

§2. Spectral sequences for exact couples

§2.1 Sketch of "rolled" exact couple

[Boa]で扱うのは unrolled exact couple だけど、その元になった idea として"rolled"の方も紹介

Def 2.1.1

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$: exact couple

$\overset{\text{def}}{\iff}$
- A, E : (graded) $\mathbb{K}$-mod
- i, j, k : (homogeneous) linear maps
- $A \xrightarrow{i} A \xrightarrow{j} E \xrightarrow{k} A \xrightarrow{i} A$: exact

Ex 2.1.2

$\{F^sC\}$: filtered cpx に対し、

$$\begin{cases} A^{st} := H^{s+t}(F^sC), & A := \bigoplus_{s,t} A^{st} \\ E^{st} := H^{s+t}\left(F^sC / F^{s+1}C\right), & E := \bigoplus_{s,t} E^{st} \end{cases}$$

と def

$$0 \to F^{s+1}C \to F^sC \to F^sC / F^{s+1}C \to 0 \text{ : exact}$$

$\downarrow H^*$

$$\begin{tikzcd} A^{s+1} \arrow[rr] & & A^s \arrow[dl] \\ & E^s \arrow[ul] & \end{tikzcd}$$: exact

$\downarrow \bigoplus_s$

$$\begin{tikzcd} A \arrow[rr] & & A \arrow[dl] \\ & E \arrow[ul] & \end{tikzcd}$$: exact couple

Lem 2.1.3

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$: exact couple に対し、
次は well-def'd!

- $A' := \operatorname{Im} i$
- $E' := H(E, jk) = \operatorname{Ker}(jk) / \operatorname{Im}(jk)$
- $i' := i|_{A'} : A' \longrightarrow A'$
- $j' : A' \longrightarrow E'$
  $i(x) \longmapsto [j(x)]$
- $k' : E' \longrightarrow A'$
  $[x] \longmapsto k(x)$

さらに、

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$: exact couple

← derived couple という。

(A, E, i, j, k) : exact couple

$\rightsquigarrow$ Define $(A_r, E_r, i_r, j_r, k_r)$ by

$$\begin{cases} \cdot\ (A_1, E_1, \cdots) := (A, E, \cdots) \\ \cdot\ (A_r, E_r, \cdots) := \text{derived couple of } (A_{r-1}, E_{r-1}, \cdots) \end{cases}$$

$\rightsquigarrow$ We have

$$E_r = H(E_{r-1}, j_{r-1}k_{r-1})$$

"spectral sequence"

Lem 2.1.4

- $A_r = \operatorname{Im}(i^{r-1} : A \to A)$
- $E_r \cong k^{-1}(\operatorname{Im} i^{r-1}) / j(\operatorname{Ker} i^{r-1})$

§2.2 Spectral sequences for unrolled exact couples

[Boa] では、収束を厳密に扱うために　[Boa, §0]

unrolled exact couple を使う

Def 2.2.1

$$\begin{tikzcd}
{} \arrow[r, dashed, "i"] & A^{s+1} \arrow[r, "i"] \arrow[dl] & & A^s \arrow[r, "i"] \arrow[dl, "j"] & & A^{s-1} \arrow[r, "i"] \arrow[dl, "j"] & & {} \\
{} & & E^s \arrow[ul, "k"] & & E^{s-1} \arrow[ul, "k"] & & {} \arrow[ul] &
\end{tikzcd}$$

: unrolled exact couple

$\overset{\Longleftrightarrow}{\text{def}}$
- A^s, E^s : graded $\mathbb{K}$-mod.
- i, j, k : homogeneous $\mathbb{K}$-linear maps
- $A^{s+1} \xrightarrow{i} A^s \xrightarrow{j} E^s \xrightarrow{k} A^{s+1} \xrightarrow{i} A^s$: exact

$E^{st} := (E^s)^{s+t}$: degree $s+t$

Rmk 2.2.2

- j, k may have non-zero degree
  (usually, $\deg(i) = 0$)
- いちいち degree を書いて E^{st} としているととても煩雑になるので、基本的にとは省略する
- 今後出てくる全ての仮定や議論は、いずれも degreewise にできる。

Rmk 2.2.3

$$0 \to \mathrm{Coker}(i : A^{s+1} \to A^s) \xrightarrow{j} E^s \xrightarrow{k} \mathrm{Ker}(i : A^{s+1} \to A^s) \to 0$$

: exact

⤷ each E^s is determined by

$$\cdots \to A^{s+1} \xrightarrow{i} A^s \xrightarrow{i} A^{s-1} \to \cdots$$ ← study!!

up to extension

Fix

$$\begin{tikzcd}
\cdots \arrow[r] & A^{s+1} \arrow[r, "i"] & A^s \arrow[r] \arrow[dl, "j"] & \cdots \\
& E^s \arrow[u, "k"] & &
\end{tikzcd}$$

: unrolled exact couple

Def 2.2.4

- For $0 \leq r < \infty$, define

$$\mathrm{Im}^r A^s := \mathrm{Im}(i \circ i \circ \cdots \circ i : A^{s+r} \longrightarrow A^s) \subset A^s$$

$$K_r A^s := \mathrm{Ker}(i \circ i \circ \cdots \circ i : A^s \longrightarrow A^{s-r}) \subset A^s$$

$(\mathrm{Im}^0 A^s = A^s, \quad K_0 A^s = 0)$ ↖[Boa, §8]の記号

- for $1 \leq r < \infty$, define

$$\begin{cases} Z^s_r := k^{-1}(\mathrm{Im}^{r-1} A^{s+1}) \subset E^s \\ B^s_r := j(K_{r-1} A^s) \subset E^s \end{cases}$$

For $r = \infty$, define　$(Q^{s+1} = \bigcap_r \mathrm{Im}^{r-1} A^{s+1})$

$$\begin{cases} Z^s_\infty := \lim_r Z^s_r = \bigcap_r Z^s_r = k^{-1}(Q^{s+1}) \\ B^s_\infty := \mathrm{colim}_r B^s_r = \bigcup_r B^s_r = j(\mathrm{Ker}\,\eta^s) \end{cases}$$

$(\eta^s : A^s \to A^{-\infty})$

- For $1 \leq r \leq \infty$, define

$$E^s_r := Z^s_\infty / B^s_\infty$$

$$0 = B^s_1 \subset B^s_2 \subset \cdots \subset B^s_\infty \subset \mathrm{Im}\, j = \mathrm{Ker}\, k \subset Z^s_\infty \subset \cdots \subset Z^s_2 \subset Z^s_1 = E^s$$

Rmk 2.2.5

Def 1.1.2 の $\tilde{Z}^{st}_r, \tilde{B}^{st}_r$ とは"全然違う"もの

こっちの方が simple (?)

Lem & Def 2.2.6

(1) For $1 \leq r < \infty$, define

$$d^s_r : E^s_r \longrightarrow E^{s+r}_r \quad : \text{well-def'd}$$
$$[x] \longmapsto [j(y)]$$

(where
$kx = i^{r-1}(y) \in \mathrm{Im}^{r-1} A^{s+1} \quad (\exists y \in A^{s+r})$)

(2) $d^s_r \circ d^{s-r}_r = 0 : E^{s-r}_r \to E^s_r \to E^{s+r}_r$

proof

(1) • $Z^s_r \longrightarrow E^{s+r}_r$

(∵ y のとり方に依らないことを示す.
$kx = i^{r-1}(y) = i^{r-1}(y')$
⤳ $y - y' \in K_{r-1}(A^{s+r})$
⤳ $jy - jy' \in j(K_{r-1} A^{s+r}) = B^{s+r}_r$)

• $Z^s_r / B^s_r \longrightarrow E^{s+r}_r$

(∵ $x = j(z)$, $z \in K_{r-1} A^s$ とすると,
$kx = kjz = 0$ ⤳ $y = 0$ とできる)

(2) は $kj = 0$ から従う.

Prop2.2.7

(1) $\mathrm{Ker}\,(d_r^s : E_r^s \to E_r^{s+r}) = Z_{r+1}^s / B_r^s$

(2) $\mathrm{Im}\,(d_r^{s-r} : E_r^{s-r} \to E_r^s) = B_{r+1}^s / B_r^s$

(see also Lem 4.4.6)

proof

(1)

($\supset$) $x \in Z_{r+1}^s$ とする.

$\exists y \in A^{s+r+1}$ s.t. $k(x) = i^r(y)$

$\rightsquigarrow d_r^s[x] = [j(i(y))] = 0$ $(\because ji = 0)$

($\subset$) $[x] \in \mathrm{Ker}\, d_r^s$ とする.

$\exists y \in A^{s+r}$ s.t. $\begin{cases} \cdot\ kx = i^{r-1}(y) \\ \cdot\ [jy] = 0 \in E_r^{s+r} \end{cases}$

$\rightsquigarrow jy \in B_r^{s+r}$

$\rightsquigarrow \exists z \in K_{r-1}A^{s+r}$ s.t. $jy = jz$

$\rightsquigarrow y - z \in \mathrm{Ker}\, j = \mathrm{Im}\, i$

$\rightsquigarrow \exists w \in A^{s+r+1}$ s.t. $y - z = iw$

$\rightsquigarrow kx = i^{r-1}(y) = i^{r-1}(z + iw) = i^r w \in \mathrm{Im}^r A^{s+1}$

$\uparrow$ $z \in \mathrm{Ker}^{r-1}$

$\rightsquigarrow x \in k^{-1}(\mathrm{Im}^r A^{s+1}) = Z_{r+1}^s$

(2)

($\supset$) $z \in B_{r+1}^s$ とする.

$\rightsquigarrow \exists y \in K_r A^s$, $z = jy$

$\rightsquigarrow i^{r-1}(y) \in \mathrm{Ker}\, i = \mathrm{Im}\, k$

$\rightsquigarrow \exists x \in E^{s-r}$ s.t. $kx = i^{r-1}(y)$

$\rightsquigarrow \begin{cases} \cdot\ x \in k^{-1}(\mathrm{Im}^{r-1} A^{s-r+1}) = Z_r^{s-r} \\ \cdot\ d_r^s[x] = [jy] = [z] \end{cases}$

($\subset$) $[x] \in E_r^{s-r}$ とする.

$d_r^{s-r}[x] = [jy]$ $(\exists y \in A^s,\ kx = i^{r-1}(y))$

$\rightsquigarrow y \in K_r A^s$

$(\because i^r(y) = ik(x) = 0)$

$\rightsquigarrow jy \in j(K_r A^s) = B_{r+1}^s$

//

Cor 2.2.8

$H^s(E_r, d_r) \cong E_{r+1}^s$

次に, spectral seq. の collapse について議論しておく

(この辺は[Boa]には書いてない
けど easy

Cor 2.2.9

(1) $d_r^s = 0 \iff Z_r^s = Z_{r+1}^s$

(2) $d_r^{s-r} = 0 \iff B_r^s = B_{r+1}^s$

proof Prop2.2.7 からすぐに分かる //

Cor 2.2.10

Fix s.

(1) **Assume**

$\exists r_0,\ \forall r \geq r_0,\ d_r^s = 0$

Then

$Z_\infty^s = \cdots = Z_{r_0+1}^s = Z_{r_0}^s$

(2) **Assume**

$\exists r_0,\ \forall r \geq r_0,\ d_r^{s-r} = 0$

Then

$B_{r_0}^s = B_{r_0+1}^s = \cdots = B_\infty^s$

proof Cor 2.2.9 からすぐ.

($Z_\infty^s = \bigcap_r Z_r^s$, $B_\infty^s = \bigcup_r B_r^s$ に注意) //

Cor 2.2.11

Assume

$\exists r_0.$ $\{E_r\}$ collapses at E_{r_0}

(i.e. $\forall r \geq r_0, \forall s,\ d_r^s = 0$)

Then

$E_{r_0} = E_{r_0+1} = \cdots = E_\infty$

collapse しなくても、

多くの場合は colim か lim で書ける

Cor 2.2.12

Fix s.

(1) **Assume**

$\exists r_0, \forall r \geq r_0, \ d_r^s = 0$

Then

$$E_\infty^s \cong \operatorname*{colim}_{r \geq r_0} E_r^s$$

where

$$\begin{tikzcd} E_{r_0}^s \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & E_{r_0+1}^s \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & \cdots \\ Z_{r_0}^s / B_{r_0}^s \arrow[r, two heads] & Z_{\boxed{r_0}}^s / B_{r_0+1}^s \arrow[r, two heads] & \cdots \end{tikzcd}$$

by assump.

(2) **Assume**

$\exists r_0, \forall r \geq r_0, \ d_r^{s-h} = 0$

Then

$$E_\infty^s \cong \lim_{r \geq r_0} E_r^s$$

where

$$\begin{tikzcd} E_{r_0}^s \arrow[d, phantom, "\shortparallel"] & E_{r_0+1}^s \arrow[l, hook'] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[l, hook'] \\ Z_{r_0}^s / B_{r_0}^s & Z_{r_0+1}^s / B_{\boxed{r_0}}^s \arrow[l, hook'] & \cdots \arrow[l, hook'] \end{tikzcd}$$

by assump.

proof

(1) $0 \to B_r^s \to \overset{\substack{Z_r^s \\ \shortparallel}}{Z_{r_0}^s} \to E_r^s \to 0$: exact $(r \geq r_0)$

$\Big\downarrow \operatorname*{colim}_r$: exact

$0 \to B_\infty^s \to \overset{\substack{Z_\infty^s \\ \shortparallel}}{Z_{r_0}^s} \to \operatorname*{colim}_r E_r^s \to 0$

$\leadsto E_\infty^s \underset{\substack{\uparrow \\ \text{def}}}{=} Z_\infty^s / B_\infty^s \cong \operatorname*{colim}_r E_r^s$

(2) $0 \to \overset{\substack{B_r^s \\ \shortparallel}}{B_{r_0}^s} \to Z_r^s \to E_r^s \to 0$: exact $(r \geq r_0)$

$\Big\downarrow \lim_r$

$0 \to \overset{\substack{B_\infty^s \\ \shortparallel}}{B_{r_0}^s} \to Z_\infty^s \to \lim_r E_r^s$

$\to \cancel{R\lim_r B_{r_0}^s} \to R\lim_r Z_r^s \to R\lim_r E_r^s \to 0$: exact

0 (∵ constant seq.)

$\leadsto E_\infty^s = Z_\infty^s / B_\infty^s \cong \lim_r E_r^s$

//

↑ Rlim については §3 で詳しく説明する。

(see Prop 3.4.3 (6))

Rmk 2.2.13

Cor 2.2.11 や Cor 2.2.12 より、

($\tilde{E}_\infty$ とは違って) E_∞ はちゃんと"それっぽい"ものになっていることが分かった。

(See Rmk 1.1.11, Ex 1.2.2)

Rmk 2.2.14

Rmk 2.2.2 の繰り返しになるが、

以上の Cor は全て degreewise にできる。

e.g.

Cor 2.2.12 (1)

Fix s, t

Assume

$\exists r_0, \forall r \geq r_0, \ d_r^{st} = 0$

Then

$$E_\infty^{st} \cong \operatorname*{colim}_{r \geq r_0} E_r^{st}$$

以上で $\{E_r\}_{1 \leq r \leq \infty}$ の構成は終わり、

[Boa] の主題は、以下の 2 つの関係を調べること：

$\begin{cases} \cdot \ E_\infty \\ \cdot \ \operatorname*{colim}_s A^s, \ \lim_s A^s, \ (R\lim_s A^s) \end{cases}$

$\shortparallel$

($H^*(\bigcup F^s C)$ for filtered cpx)

0 になっていてほしいもの

§2.3 Comparison of two constructions:
filtered complexes and exact couples

Fix

$\{F^sC\}$: filtered cpx

(bounded とかは何も仮定しない)

Define an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[rr] & & A^{s+1} \arrow[rr] \arrow[dl] & & A^s \arrow[rr] \arrow[dl] & & A^{s-1} \arrow[rr] \arrow[dl] & & \cdots \\
& {} & & E^s \arrow[ul] & & E^{s-1} \arrow[ul] & & {} \arrow[ul] &
\end{tikzcd}$$

by

$$\begin{cases} \cdot\ A^s := H^*(F^sC) \\ \cdot\ E^s := H^*(F^sC/F^{s+1}C) \end{cases}$$

Then we have two spectral sequences

$$\begin{cases} \cdot\ \{\tilde{E}_r^{st}\} : \text{defined in §1.1} \\ \cdot\ \{E_r^{st}\} : \text{defined in §2.2} \end{cases}$$

$\hookleftarrow E_r^{st} = (E_r^s)^{s+t}$: degree $(s+t)$

これらを比較するために，

E_r^{st} を（E_1^s ではなく）C の subquotient として書く．

Notation

煩雑さを回避するため，証明中とかでは

$$\begin{cases} \cdot\ \text{total degree (or } t\text{) を省略} \\ \cdot\ F^s := F^sC \end{cases}$$

とする．

Lem2.3.1

$$E_1^{st} \cong \frac{F^sC^{s+t} \cap d^{-1}(F^{s+1}C^{s+t+1})}{F^{s+1}C^{s+t} + d(F^sC^{s+t-1})}$$

proof

$E_1^s = H^*(F^s/F^{s+1})$

$$\begin{cases} \mathrm{Ker}\left(d : F^s/F^{s+1} \to F^s/F^{s+1}\right) = F^s \cap d^{-1}(F^{s+1})\big/F^{s+1} \\ \mathrm{Im}\left(d : F^s/F^{s+1} \to F^s/F^{s+1}\right) = F^{s+1} + d(F^s)\big/F^{s+1} \end{cases}$$

//

また，Lem2.3.1 の下で exact couple は次のように書ける：

$$\begin{tikzcd}
{} \arrow[r] & A^{s+1} \arrow[r, "i"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & A^s \arrow[r, "j"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & E^s \arrow[r, "k"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & A^{s+1} \arrow[r] \arrow[d, equal] & {} \\
& \frac{F^{s+1} \cap \mathrm{Ker}\,d}{d(F^{s+1})} \arrow[r] & \frac{F^s \cap \mathrm{Ker}\,d}{d(F^s)} \arrow[r] & \frac{F^s \cap d^{-1}(F^{s+1})}{F^{s+1} + d(F^s)} \arrow[r] & \frac{F^{s+1} \cap \mathrm{Ker}\,d}{d(F^{s+1})} & \\
& {[x]} \arrow[r, mapsto] & {[x]} \arrow[r, mapsto] & {[x]} \arrow[r, mapsto] & {[dx]} &
\end{tikzcd}$$

Prp2.3.2

Lem2.3.1 の isom の下で，$1 \leq r < \infty$ に対し

$$Z_r^{st} = \frac{F^{s+1}C^{s+t} + \tilde{Z}_r^{st}}{F^{s+1}C^{s+t} + d(F^sC^{s+t-1})}$$

$$B_r^{st} = \frac{F^{s+1}C^{s+t} + \tilde{B}_{r-1}^{st}}{F^{s+1}C^{s+t} + d(F^sC^{s+t-1})}$$

proof $\cdot\ Z_r^s = k^{-1}(\mathrm{Im}(A^{s+r} \to A^{s+1})) \subset E_1^s$

$(Z_r^s \supset \mathrm{RHS})$

$\forall [x] \in \mathrm{RHS}$ をとる．

$x \in \tilde{Z}_r^s$ としてよい．（☺ F^{s+1} は分母にも含まれている）

$\rightsquigarrow dx \in F^{s+r}$

よって，

$k[x] = [dx] \in \mathrm{Im}(\overset{\overset{H^*(F^{s+r})}{\shortparallel}}{A^{s+r}} \to A^{s+1})$

$\rightsquigarrow [x] \in Z_r^s$

$(Z_r^s \subset \mathrm{RHS})$

$[x] \in Z_r^s$ とすると

$\exists [y] \in A^{s+r}$ s.t. $k[x] = i^{r-1}[y] \in A^{s+1}$

i.e. $\exists y \in F^{s+r} \cap \mathrm{Ker}\,d$ s.t. $\underset{\substack{\shortparallel \\ z}}{dx - y} \in d(\underset{\ni z}{F^{s+1}})$

$\rightsquigarrow x - z \in F^s \cap d^{-1}(y) \subset \underset{\substack{\shortparallel \\ \tilde{Z}_r^s}}{F^s \cap d^{-1}(F^{s+r})}$

$\rightsquigarrow x \in F^{s+1} + \tilde{Z}_r^s$

$\cdot\ B_r^s = j(\mathrm{Ker}(A^s \to A^{s-r+1}))$

$(B_r^s \supset \mathrm{RHS})$

$\forall [x] \in \mathrm{RHS}$ をとる．$x \in \tilde{B}_r^s$ としてよい

$\rightsquigarrow \exists z \in F^{s-r+1}$ s.t. $x = dz$ $(\in F^s)$

$\rightsquigarrow [x]_{A^s} \in \mathrm{Ker}(A^s \to A^{s-r+1})$

$\rightsquigarrow [x]_{E^s} \in B_r^s$

$(B_r^s \subset \mathrm{RHS})$

$[x] \in B_r^s$ とすると，

$\exists [y] \in \mathrm{Ker}(A^s \to A^{s-r+1})$ s.t. $j[y] = [x] \in E^s$

i.e. $\exists y \in F^s \cap \mathrm{Ker}\,d$ s.t. $\begin{cases} (1)\ y \in d(F^{s-r+1}) \\ (2)\ y - x \in F^{s+1} + d(F^s) \end{cases}$

(1) $\Leftrightarrow \exists z \in F^{s-r+1}$ s.t. $y = dz$ $(\in F^s)$

(2) $\Leftrightarrow \exists w \in F^{s+1}, \exists u \in F^s$ s.t. $y - x = w + d(u)$

$\rightsquigarrow x = \underset{\substack{\cap \\ F^{s+1}}}{-w} + \underset{\substack{\cap \\ F^s \cap d(F^{s-r+1}) = \tilde{B}_{r-1}^s}}{dz} - \underset{\in d(F^s)}{du}$ ← (☺ $F^s \subset F^{s-r+1}$ ← $r \geq 1$)

//

Cor 2.3.3

For $1 \leq {}^\forall r < \infty$, we have

$$E^{st}_r \cong \tilde{E}^{st}_r \quad (\text{compatible with } d_r)$$

proof Prop 2.3.2 より

$$E^s_r = Z^s_r / B^s_r = \frac{(F^{s+1}+\tilde{Z}^s_r) / (F^{s+1}+d(F^s))}{(F^{s+1}+\tilde{B}^s_{r-1}) / (F^{s+1}+d(F^s))}$$

$$\cong \frac{F^{s+1}+\tilde{Z}^s_r}{F^{s+1}+\tilde{B}^s_{r-1}}$$

$$\cong \frac{\tilde{Z}^s_r}{\tilde{Z}^s_r \cap (F^{s+1}+\tilde{B}^s_{r-1})} \quad \left(∵ \ \frac{A+C}{A+B} \cong \frac{C}{C\cap(A+B)} \right)$$

$$= \frac{\tilde{Z}^s_r}{\underbrace{(\tilde{Z}^s_r \cap F^{s+1})}_{\overset{\shortparallel}{\tilde{Z}^{s+1}_r}} + \tilde{B}^s_{r-1}} \quad \left(∵ \ C\cap(A+B) \supset (C\cap A)+(C\cap B) \right.$$
$\left. C \supset B \Rightarrow \text{等号成立.} \right)$

d_r については、いずれも d (in C) から induce されたもの なので明らか.

//

以上より、($r=\infty$ のあたりを除いて)

2つの s.s. が一致することが分かった

Rmk 2.3.4

- $Z^s_\infty \neq \frac{F^{s+1}+\tilde{Z}^s_\infty}{F^{s+1}+d(F^s)}$

  $\left(∵ \ \bigcap_r (F^{s+1}+\tilde{Z}^s_r) \neq F^{s+1} + \bigcap_r \tilde{Z}^s_r \text{ in general} \right.$
  $\left. \text{see Lem 2.3.6} \right)$ ← 一応 $R\lim_r(F^{s+1}+d(F^s))=0$ も大事.

- $B^s_\infty = \frac{F^{s+1}+\tilde{B}^s_\infty}{F^{s+1}+d(F^s)}$

  (∵ ・colim: exact.
  ・$\bigcup_r (F^{s+1}+\tilde{B}^s_r) = F^{s+1} + \bigcup_r \tilde{B}^s_r$)

$\rightsquigarrow E^s_\infty \neq \tilde{E}^s_\infty$ (c.f. Rmk 1.1.3)

Rmk 2.3.5

以上をふまえると、(bounded でない場合には)

$\tilde{E}_\infty$ ではなく E_∞ を考えた方が良い.

- filtration の情報がなくても、
  $\{E_r\}$ さえ分かれば E_∞ は決まる (Cor 2.2.11, 2.2.12)
- $\tilde{E}_\infty$ は $\{E_r\}$ だけでは決まらない (Ex 1.2.2)
  かといって $F^sH(C)/F^{s+1}H(C)$ と isom というわけでもなく.
  中途半端な感じ.

Lem 2.3.6 (余談)

M: module

$N \subset M$: submodule

$\cdots \subset M^{s+1} \subset M^s \subset \cdots \subset M^0 = M$: seq of submodules

Then

(1) $N + \bigcap_s M^s \subset \bigcap_s (N+M^s) \quad (\subset M)$

(2) $\exists M, N, \{M^s\}$ s.t.
(1) is proper subset (i.e. $\subsetneq$)

proof (1) 明らか.

(2) $M := \mathbb{K}[x]$

$N := \{ f \in \mathbb{K}[x] \mid f(1) = 0 \}$

$M^s := x^s \mathbb{K}[x]$

Then we have

- ${}^\forall s$, $N + M^s = M$

  $\left(∵ \ {}^\forall f \in M \text{ に対し}, \right.$
  $\left. f = \underset{\underset{N}{\cap}}{(f - x^s f)} + \underset{\underset{M^s}{\cap}}{x^s f} \quad \text{for } {}^\forall s \right)$

- $\bigcap_s M^s = 0$

Hence

$$N = N + \bigcap_s M^s \subsetneq \bigcap_s (N+M^s) = M$$

//

(↑ M: fin. gen にしたいなら:
$\mathbb{K} = \mathbb{Z}$ とし、$n \geq 2 \in \mathbb{Z}$ を fix.

$$\begin{cases} M := \mathbb{Z} \\ N := (n-1)\mathbb{Z} \\ M^s := n^s \mathbb{Z} \end{cases}$$

$\rightsquigarrow \begin{cases} \cdot \ {}^\forall s, \ N + M^s = \mathbb{Z} \\ \cdot \ \bigcap_s M^s = 0 \end{cases}$)

§3. Tools: limits and colimits [Boa. Part I]

Def

- $\{A^s\} = (\cdots \to A^{s+1} \xrightarrow{i} A^s \to \cdots)$

  $\{B^s\} = (\cdots \to B^{s+1} \xrightarrow{i} B^s \to \cdots)$

  $f : \{A^s\} \longrightarrow \{B^s\}$: morph of seq's

  $\underset{\text{def}}{\Longleftrightarrow}$ $f = \{f^s : A^s \to B^s\}_s$ s.t. commutes with i

- $0 \to \{A^s\} \xrightarrow{f} \{B^s\} \xrightarrow{g} \{C^s\} \to 0$: exact seq of seqs

  $\underset{\text{def}}{\Longleftrightarrow}$ · f, g : morph of seq's

  · $\forall s.\ 0 \to A^s \xrightarrow{f^s} B^s \xrightarrow{g^s} C^s \to 0$: exact

§3.1 Colimits (easy)

Def3.1.1

$\cdots \longrightarrow A^{s+1} \xrightarrow{i} A^s \longrightarrow \cdots$

- $1-i : \bigoplus_s A^s \longrightarrow \bigoplus_s A^s$

  $(a^s) \longmapsto (a^s - i(a^{s+1}))_s$

- $\operatorname{colim}_s A^s = A^{-\infty} := \operatorname{Coker}(1-i)$

$$\begin{tikzcd}
 & A^s \arrow[dl, "\eta^s"'] \\
\operatorname{Coker}(1-i) \arrow[r, phantom, "\cong"] & A^{s+1} \arrow[u, "i"'] \arrow[l, "\eta^{s+1}"]
\end{tikzcd}$$

Ex3.1.2

M: module

$\cdots \subset F^{s+1}M \subset F^sM \subset \cdots \subset M$: seq of submods

Then

$\operatorname{colim}_s F^sM \cong \bigcup_s F^sM$

Lem3.1.3

(1) $\forall x \in A^{-\infty}$ に対し、

$\exists s, \exists a \in A^s$ s.t. $\eta^s : A^s \longrightarrow A^{-\infty}$

$a \longmapsto x$

(2) $\eta^s(a), \eta^t(b) \in A^{-\infty}$ について.

$\eta^s(a) = \eta^t(b)$

$\Longleftrightarrow \exists n < s, t$ s.t. $i^{s-n}(a) = i^{t-n}(b) \in A^n$

proof 略. //

この辺の詳細は

§3.2を参照

see also Prop3.2.14

Prop3.1.4

(1) $1-i$ is inj ($\leftrightarrow$ No need to consider $\operatorname{Ker}(1-i)$)

(2) colim is exact

(i.e. $0 \to \{A^s\} \to \{B^s\} \to \{C^s\} \to 0$, exact seq of seq's

$\Rightarrow 0 \to A^{-\infty} \to B^{-\infty} \to C^{-\infty} \to 0$: exact)

proof

(1) $(a^s)_s \in \operatorname{Ker}(1-i)$ とする.

$S_0 := \widetilde{\max}\{s \mid a^s \neq 0\}$

⊕だからとれる

$\hookrightarrow$ $0 = a^{s_0} - i(\underset{\overset{\shortparallel}{0}}{\widetilde{a^{s_0+1}}}) = a^{s_0}$

$\hookrightarrow$ induction により、$\forall s < s_0$. $a^s = 0$ が分かる.

(2)

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus A^s \arrow[r] \arrow[d, "1-i"] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus B^s \arrow[r] \arrow[d, "1-i"] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus C^s \arrow[r] \arrow[d, "1-i"] & 0 & \text{: exact} \\
0 \arrow[r] & \bigoplus A^s \arrow[r] & \bigoplus B^s \arrow[r] & \bigoplus C^s \arrow[r] & 0 & \text{: exact}
\end{tikzcd}$$

snake lemma + (1) //

colim が A^s for small s だけで決まることを示したい.

Def3.1.5

$\{A^s\} \underset{-\infty}{\cong} \{B^s\}$: isom around $-\infty$

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists s_0, \exists f = \{f^s\}_{s \le s_0}$ s.t. $\forall s \le s_0$

$f^s : A^s \xrightarrow{\cong} B^s$: isom

Lem3.1.6

(1) $\{A^s\} \underset{-\infty}{\cong} 0 \Rightarrow A^{-\infty} = 0$

(2) $f : \{A^s\} \to \{B^s\}$: morph of seq's

$\exists s_0, \forall s \le s_0, f^s$: isom

Then

$f^{-\infty} : A^{-\infty} \xrightarrow{\cong} B^{-\infty}$: isom

proof (1) Lem3.1.3(1) を使う

(2) $\operatorname{Ker} f, \operatorname{Coker} f \underset{-\infty}{\cong} 0$ より ok. //

Prop3.1.7

$\{A^s\} \underset{-\infty}{\cong} \{B^s\} \Rightarrow A^{-\infty} \cong B^{-\infty}$

proof

$$\begin{tikzcd}
 & \arrow[r] & B^{s_0+2} \arrow[r] & B^{s_0+1} \arrow[r] & B^{s_0} \arrow[r] & B^{s_0-1} \arrow[r] & \cdots & = & \{B^s\} \\
g & \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] \arrow[ur, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & A^{s_0} \arrow[r] \arrow[u, "\cong"] \arrow[d, "="] & A^{s_0-1} \arrow[r] \arrow[u, "\cong"] \arrow[d, "="] & \cdots & = & ① \arrow[u] \arrow[d] \\
f & \arrow[r] & A^{s_0+2} \arrow[r] & A^{s_0+1} \arrow[r] & A^{s_0} \arrow[r] & A^{s_0-1} \arrow[r] & \cdots & = & \{A^s\}
\end{tikzcd}$$

//

§3.2 Limits [Boa, §1]

Def 3.2.1

- $1-i : \prod_s A^s \longrightarrow \prod_s A^s$

  $(a^s) \longmapsto (a^s - i(a^{s+1}))_s$
- $\lim_s A^s = A^\infty := \mathrm{Ker}(1-i)$

  $R\lim_s A^s = RA^\infty := \mathrm{Coker}(1-i)$

$$\begin{tikzcd} & A^s \\ A^\infty \arrow[ur, "\varepsilon^s"] \arrow[dr, "\varepsilon^{s+1}"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & A^{s+1} \arrow[uu, "i"'] \end{tikzcd}$$

$(a^s) \in A^\infty \iff \forall s,\ a^s = i(a^{s+1})$

Policy

Never to mention A^∞ without also introducing RA^∞

Ex 3.2.2

M: module

$\cdots \subset F^{s+1}M \subset F^sM \subset \cdots$

Then

$$\lim_s F^sM = \bigcap_s F^sM$$

Prop 3.2.3 [Boa, Thm 1.4]

$0 \to \{A^s\} \to \{B^s\} \to \{C^s\} \to 0$: exact seq of seq's

Then

$0 \to A^\infty \to B^\infty \to C^\infty$

$\to RA^\infty \to RB^\infty \to RC^\infty \to 0$: exact

proof

apply snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & \prod_s A^s \arrow[r] \arrow[d, "1-i"] & \prod_s B^s \arrow[r] \arrow[d, "1-i"] & \prod_s C^s \arrow[r] \arrow[d, "1-i"] & 0 \ :\ \text{exact} \\ 0 \arrow[r] & \prod_s A^s \arrow[r] & \prod_s B^s \arrow[r] & \prod_s C^s \arrow[r] & 0 \ :\ \text{exact} \end{tikzcd}$$

//

Cor 3.2.4 [Boa, Cor 1.6]

$\{A^s\} \to \{B^s\} \to \{C^s\} \to 0$: exact

Assume $RA^\infty = 0$

Then

$B^\infty \twoheadrightarrow C^\infty$: surj

proof

$K^s := \mathrm{Ker}(A^s \to B^s)$, $I^s := \mathrm{Im}(A^s \to B^s)$

$\to$ $RI^\infty = 0$

(☺ $0 \to \{K^s\} \to \{A^s\} \to \{I^s\} \to 0$: exact

↪ $\cdots \to RK^\infty \to \underset{0}{\cancel{RA^\infty}} \to RI^\infty \to 0$: exact)

$0 \to \{I^s\} \to \{B^s\} \to \{C^s\} \to 0$: exact

↪ $\cdots \to B^\infty \to C^\infty$

$\to \underset{0}{\cancel{RI^\infty}} \to \cdots$: exact

//

Prop 3.2.5 [Boa, Prop 1.8]

Assume

$\exists s_0,\ \forall s \geq s_0,\ i : A^{s+1} \twoheadrightarrow A^s$: surj.

Then

(a) $\forall s \geq s_0,\ \varepsilon^s : A^\infty \twoheadrightarrow A^s$: surj.

(b) $RA^\infty = 0$

proof

(a) $\forall a \in A^s$ とる.

Enough to show:

$\exists (a^t)_t \in \prod A^t$ s.t. $\begin{cases} \cdot\ a^t = i(a^{t+1}) \\ \cdot\ a^s = a \end{cases}$

- Define $a^t := i^{s-t}(a)$ for $t \leq s$
- $a^s \xleftarrow[\text{surj}]{i} {}^\exists a^{s+1} \xleftarrow[\text{surj}]{i} {}^\exists a^{s+2} \xleftarrow[\text{surj}]{i} \cdots\cdots$

(b) $\forall (a^s) \in \prod A^s$ とる.

Enough to show:

${}^\exists (b^s) \in \prod A^s$ s.t. $(1-i)((b^s)) = (a^s)$

(i.e. $b^s - i(b^{s+1}) = a^s$)

Define

- $b^{s_0} := 0$
- For $\forall s < s_0$, define inductively by

  $b^s := a^s + i(b^{s+1})$
- For $\forall s > s_0$, choose b^s inductively by

  $i(b^s) = b^{s-1} - a^{s-1}$

//

Cor 3.2.6 [Boa, Cor 1.9]

Assume

- $\forall s,\ i : A^{s+1} \twoheadrightarrow A^s$: surj
- $A^\infty = 0$

Then

$\forall s,\ A^s = 0$

Ex3.2.7

p : prime $\quad (\mathbb{K}=\mathbb{Z})$

Consider the sequence

$$\{A^s\}=\left(\cdots\to\mathbb{Z}\xrightarrow{p\times}\mathbb{Z}\xrightarrow{p\times}\mathbb{Z}\to0\to0\to\cdots\right)$$

Then we have

$$A^\infty=0,\quad RA^\infty=\hat{\mathbb{Z}}_p/\mathbb{Z}$$

where

$$\hat{\mathbb{Z}}_p:=\lim_s\left(\cdots\to\mathbb{Z}/p^s\to\mathbb{Z}/p^{s-1}\to\cdots\to\mathbb{Z}/p\to0\right)$$

uncountable ab. group

∵ $(A^\infty=0$ follows from Ex3.2.2)

Consider the exact sequence

$$\begin{tikzcd}
 & 0 \arrow[d] \\
\{A^s\}= & (\cdots\to\mathbb{Z} \arrow[r, "p\times"] \arrow[d, "p^3\times"] & \mathbb{Z} \arrow[r, "p\times"] \arrow[d, "p^2\times"] & \mathbb{Z} \arrow[r] \arrow[d, "p\times"] & 0\to\cdots)\\
\{B^s\}= & (\cdots\to\mathbb{Z} \arrow[r, "id"] \arrow[d] & \mathbb{Z} \arrow[r, "id"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "prj"] & 0\to\cdots) \quad \text{constant seq.}\\
\{C^s\}= & (\cdots\to\mathbb{Z}/p^3 \arrow[r, "prj"] & \mathbb{Z}/p^2 \arrow[r, "prj"] & \mathbb{Z}/p \arrow[r] & 0\to\cdots)\\
 & 0
\end{tikzcd}$$

Now we have

$$0\to A^\infty\to \overset{\mathbb{Z}}{\overset{\shortparallel}{B^\infty}}\to \overset{\hat{\mathbb{Z}}_p}{\overset{\shortparallel}{C^\infty}} \leftarrow \text{by def}$$

$$\to RA^\infty\to \underset{0}{\underset{\shortparallel}{RB^\infty}}\to RC^\infty\to0 \quad : \text{exact}$$

$\leftarrow$ Prop3.2.5(b)

Ex3.2.8

$$A^s:=\bigoplus_{0\le n\le s}\mathbb{K}$$

$$\begin{tikzcd}
i: A^{s+1} \arrow[r] \arrow[d, equal] & A^s \arrow[d, equal]\\
\bigoplus_{0\le n\le s+1}\mathbb{K} \arrow[r, "prj"] & \bigoplus_{0\le n\le s}\mathbb{K}
\end{tikzcd}$$

$$\begin{tikzcd}
\to A^2 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, equal] & A^0 \arrow[d, equal]\\
(\cdots\to\mathbb{K}^3 \arrow[r] & \mathbb{K}^2 \arrow[r] & \mathbb{K})
\end{tikzcd}$$

Then

$$A^\infty=\prod_{n\ge0}\mathbb{K},\quad RA^\infty=0$$

∵ A^∞ は by def で計算

$RA^\infty=0$ は Prop3.2.5 (b) より.

Ex3.2.9

$$B^s:=\bigoplus_{n\ge s}\mathbb{K}$$

$$\begin{tikzcd}
i: B^{s+1} \arrow[r] \arrow[d, equal] & B^s \arrow[d, equal]\\
\bigoplus_{n\ge s+1}\mathbb{K} \arrow[r, "incl"] & \bigoplus_{n\ge s}\mathbb{K}
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & B^2 \arrow[r] \arrow[d, equal] & B^1 \arrow[r] \arrow[d, equal] & B^0 \arrow[d, equal]\\
\Bigg(\cdots & \begin{matrix} \\ \\ \mathbb{K}\\ \oplus\\ \vdots\end{matrix} \arrow[r] & \begin{matrix} \\ \mathbb{K}\\ \oplus\\ \mathbb{K}\\ \oplus\\ \vdots\end{matrix} \arrow[r] & \begin{matrix}\mathbb{K}\\ \oplus\\ \mathbb{K}\\ \oplus\\ \mathbb{K}\\ \oplus\\ \vdots\end{matrix}\Bigg)
\end{tikzcd}$$

Then

$$B^\infty=0,\quad RB^\infty=\prod_{n\ge0}\mathbb{K}\Big/\bigoplus_{n\ge0}\mathbb{K}$$

∵ Define

- $\{A^s\}$ as in Ex3.2.8
- $\{C^s\}$: constant seq at $\bigoplus_{n\ge0}\mathbb{K}$

Then we have

$$0\to\{B^s\}\to\{C^s\}\to\{A^s\}\to0 \quad : \text{exact}$$

$$0\to B^\infty\to \overset{\bigoplus\mathbb{K}}{\overset{\shortparallel}{C^\infty}}\to \overset{\prod\mathbb{K}}{\overset{\shortparallel}{A^\infty}}\to$$

$$\to RB^\infty\to \underset{0}{\underset{\shortparallel}{RC^\infty}}\to RA^\infty\to0 \quad : \text{exact}$$

$\leftarrow$ Prop3.2.5(b)

Prop3.2.10 [Boa, Prop1.10]

Λ : set

$A(\lambda)=\{A(\lambda)^s\}_s$: seq. (for $\lambda\in\Lambda$)

Define

$$A:=\prod_\lambda A(\lambda)\quad \left(\text{i.e. } A^s=\prod_\lambda A(\lambda)^s\right)$$

Then

$$\lim_s A^s=\prod_\lambda\lim_s A(\lambda)^s$$

$$R\lim_s A^s=\prod_\lambda R\lim_s A(\lambda)^s$$

proof

$$\prod_\lambda\prod_s=\prod_s\prod_\lambda$$

//

lim, Rlim が A^s for large s だけで決まることを示したい。

Def3.2.11

$\{A^s\} \underset{\infty}{\cong} \{B^s\}$: isom around ∞

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists s_0$, $\exists f = \{f^s\}_{s \geq s_0}$ s.t. $\forall s \geq s_0$

$f^s : A^s \xrightarrow{\cong} B^s$: isom

Lem3.2.12

(1) $\{A^s\} \underset{\infty}{\cong} 0 \Rightarrow A^\infty = RA^\infty = 0$

(2) $f : \{A^s\} \to \{B^s\}$: morph of seq's

$\exists s_0$, $\forall s \geq s_0$, f^s : isom

Then

$$\left.\begin{array}{l} f^\infty : A^\infty \xrightarrow{\cong} B^\infty \\ Rf^\infty : RA^\infty \xrightarrow{\cong} RB^\infty \end{array}\right\} : \text{isom}$$

proof

(1) $RA^\infty = 0$ は Prop3.2.5(b) より ok.

$(a^s) \in A^\infty$ とすると.

$$\begin{tikzcd}
\cdots \arrow[r, mapsto] & 0 \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & a^{s_0-1} \arrow[r, mapsto] \arrow[d, phantom, "\|"] & a^{s_0-1} \arrow[r, mapsto] \arrow[d, phantom, "\|"] & \cdots \\
 & A^{s_0}=0 \arrow[r, squiggly] & 0 \arrow[r, squiggly] & 0 \arrow[r, squiggly] & \cdots
\end{tikzcd}$$

$\rightsquigarrow$ $\forall s$, $a^s = 0$

(2) Define seq's $\{K^s\}, \{I^s\}, \{C^s\}$ by

$K^s := \mathrm{Ker} f^s$, $I^s := \mathrm{Im} f^s$, $C^s := \mathrm{Coker} f^s$

$$\rightsquigarrow \begin{cases} 0 \to \{K^s\} \to \{A^s\} \to \{I^s\} \to 0 & : \text{exact} \\ 0 \to \{I^s\} \to \{B^s\} \to \{C^s\} \to 0 & : \text{exact} \end{cases}$$

By assump.

$\{K^s\} \underset{\infty}{\cong} 0$, $\{C^s\} \underset{\infty}{\cong} 0$

$\overset{(1)}{\rightsquigarrow}$ $K^\infty = RK^\infty = 0$, $C^\infty = RC^\infty = 0$

Now, long exact seq implies

$$\begin{cases} A^\infty \cong I^\infty \cong B^\infty \\ RA^\infty \cong RI^\infty \cong RB^\infty \end{cases} \quad \text{(by canonical maps)}$$

//

Prop3.2.13

$$\{A^s\} \underset{\infty}{\cong} \{B^s\} \Rightarrow \begin{cases} A^\infty \cong B^\infty \\ RA^\infty \cong RB^\infty \end{cases}$$

proof

Take

$f = \{f^s : A^s \xrightarrow{\cong} B^s\}_{s \geq s_0}$

Define $\{\overline{B}^s\}$ by

$$\overline{B}^s := \begin{cases} B^s & (s \geq s_0) \\ 0 & (s < s_0) \end{cases}$$

Then we have

$$\begin{tikzcd}
\{A^s\} \arrow[d] & = (\cdots \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & A^{s_0+1} \arrow[r] \arrow[d, "f^{s_0+1}"] \arrow[dr, phantom, "\circlearrowleft"] & A^{s_0} \arrow[r] \arrow[d, "f^{s_0}"] \arrow[dr, phantom, "\circlearrowleft"] & A^{s_0-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \cdots) \\
\{\overline{B}^s\} & = (\cdots \arrow[r] & B^{s_0+1} \arrow[r] & B^{s_0} \arrow[r] & 0 \arrow[r] & 0 \cdots) \\
\{B^s\} \arrow[u] & = (\cdots \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & B^{s_0+1} \arrow[r] \arrow[u, "\mathrm{id}"] \arrow[ur, phantom, "\circlearrowleft"] & B^{s_0} \arrow[r] \arrow[u, "\mathrm{id}"'] \arrow[ur, phantom, "\circlearrowleft"] & B^{s_0-1} \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & \cdots)
\end{tikzcd}$$

By Lem3.2.12,

$A^\infty \xrightarrow{\cong} \overline{B}^\infty \xleftarrow{\cong} B^\infty$

$RA^\infty \xrightarrow{\cong} R\overline{B}^\infty \xleftarrow{\cong} RB^\infty$

//

最後に、(c) limit が index の shift で変わらないことを示す.

Prop 3.2.14

Fix $k \geq 0$

Then $i^k : A^{s+k} \to A^s$ induces ← これが非自明.

(1) $\operatorname{colim}_s A^{s+k} \xrightarrow{\cong} \operatorname{colim}_s A^s$

(2) $\lim_s A^{s+k} \xrightarrow{\cong} \lim_s A^s$

$R\lim_s A^{s+k} \xrightarrow{\cong} R\lim_s A^s$

もちろん inj, surj を元をとって直接示しても良いが
それは少し面倒なので工夫する

Lem 3.2.15 [Bua, Lem 1.7]

$$\begin{tikzcd}
 & & K^3 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & K^1 \arrow[d] & \\
A^5 \arrow[r] \arrow[d, "g^5"] & A^4 \arrow[r] \arrow[d, "g^4"] & A^3 \arrow[r] \arrow[d, "g^3"] & A^2 \arrow[r] \arrow[d, "g^2"] & A^1 & : \text{exact} \\
B^5 \arrow[r] \arrow[d] & B^4 \arrow[r] \arrow[d] & B^3 \arrow[r] \arrow[d] & B^2 \arrow[r] & B^1 & : \text{exact} \\
C^5 \arrow[r] & C^4 \arrow[r] & C^3 & & &
\end{tikzcd}$$

$K^s := \operatorname{Ker} g^s$, $C^s := \operatorname{Coker} g^s$

Then

- There is a canonical isom
  $H(K^3 \to K^2 \to K^1) \xleftrightarrow{\cong} H(C^5 \to C^4 \to C^3)$
  ↖ homology at K^2 ↖ homology at C^4
- $K^3 \to K^2 \to K^1$: exact
  $\Leftrightarrow C^5 \to C^4 \to C^3$: exact

diagram chasing をするだけなので. proof は省略.

ちなみに、

- snake lemma
- 5-lemma

は Lem 3.2.15 から従う.

proof of Prop 3.2.14

いずれも $k=1$ の case を示せば十分.

(☺ $\operatorname{colim}_s A^{s+k} \xrightarrow{\cong} \operatorname{colim}_s A^{s+k-1} \xrightarrow{\cong} \cdots \xrightarrow{\cong} \operatorname{colim}_s A^s$)

Define

$$\begin{cases} K^{s+1} := \operatorname{Ker}(i : A^{s+1} \to A^s) \\ C^s := \operatorname{Coker}(i : A^{s+1} \to A^s) \end{cases}$$

(2)

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \lim_s A^{s+1} \arrow[r] \arrow[d] & \lim_s A^s \arrow[r] \arrow[d] & 0 \arrow[d] & \leftarrow \text{Ker} \\
0 \arrow[r] & \prod_s K^{s+1} \arrow[r] \arrow[d, "1", "\cong"'] & \prod_s A^{s+1} \arrow[r, "i"] \arrow[d, "1-i"] & \prod_s A^s \arrow[r] \arrow[d, "1-i"] & \prod_s C^s \arrow[r] \arrow[d, "1", "\cong"'] & 0 : \text{exact} \quad \leftarrow \text{similar to } K^s \\
0 \arrow[r] & \prod_s K^{s+1} \arrow[r] \arrow[d] & \prod_s A^{s+1} \arrow[r, "i"] \arrow[d] & \prod_s A^s \arrow[r] \arrow[d] & \prod_s C^s \arrow[r] \arrow[d] & 0 : \text{exact} \\
 & 0 \arrow[r] & R\lim_s A^{s+1} \arrow[r] & R\lim_s A^s \arrow[r] & 0 & \leftarrow \text{Coker}
\end{tikzcd}$$

(☹ $i=0$ on $K^{s+1} = \operatorname{Ker} i$)

← $k \geq 2$ だとここで詰まる ($\operatorname{Ker} i^2$)

Lem 3.2.15 を4回使えば

Ker の段と Coker の段が exact であることが分かる.

(1) 同様なので略.

//

§3.3 Cohomology and (co)limits

Lem3.3.1[DK, §2.6 Exercise 29]

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "i"] & A \arrow[r, "f"] & B \arrow[r, "p"] \arrow[dl, "g"] & F \arrow[r] & 0 \\ & & C \arrow[ul, "h"] \arrow[u, phantom, "\text{exact}"] & & & \end{tikzcd}$$: exact

Then

$$0 \to F \xrightarrow{g \circ p^{-1}} C \xrightarrow{i^{-1} \circ h} E \to 0$$

is well-def'd and exact

proof diagram chasing //

Thm3.3.2

$$\cdots \to C^{s+1} \xrightarrow{i} C^s \xrightarrow{i} \cdots$$: seq of cpx's

(i.e. $C^s = \{(C^s)^n\}_{n \in \mathbb{Z}}$
$d : (C^s)^n \to (C^s)^{n+1}$)

$\rightsquigarrow$

$$\cdots \to H(C^{s+1}) \xrightarrow{H(i)} H(C^s) \xrightarrow{H(i)} \cdots$$: seq of mod's

Then

(1) $\operatorname{colim}_s H^n(C^s) \xrightarrow{\cong} H^n(\operatorname{colim}_s C^s)$

(2) Assume $R\varprojlim_s C^s = 0$

Then

$$0 \to R\varprojlim_s H^{n-1}(C^s) \to H^n(\varprojlim_s C^s) \to \varprojlim_s H^n(C^s) \to 0$$

exact

proof

(1) We have

(a) $0 \to \bigoplus_s C^s \xrightarrow{1-i} \bigoplus_s C^s \longrightarrow \operatorname{colim}_s C^s \to 0$: exact

(b) $0 \to \bigoplus_s H(C^s) \xrightarrow{1-H(i)} \bigoplus_s H(C^s) \longrightarrow \operatorname{colim}_s H(C^s) \to 0$: exact

By (a), $\parallel \quad \circlearrowleft \quad \parallel$

$$\begin{tikzcd} H(\bigoplus_s C^s) \arrow[rr, "H(1-i)"] & & H(\bigoplus_s C^s) \arrow[dl] \\ & H(\operatorname{colim}_s C^s) \arrow[ul, "\deg+1"] \arrow[u, phantom, "\text{exact}"] & \end{tikzcd}$$

: long exact seq.

Hence we have

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \bigoplus_s H(C^s) \arrow[r] & \bigoplus_s H(C^s) \arrow[r] \arrow[dl] & \operatorname{colim}_s H(C^s) \arrow[r] & 0 \\ & & H(\operatorname{colim}_s C^s) \arrow[u] \arrow[u, phantom, "\text{exact}"] & & & \end{tikzcd}$$

exact

$\rightsquigarrow$ Lem3.3.1 implies the isom.

(2) We have

since $R\varprojlim_s C^s = 0$

(c) $0 \to \varprojlim_s C^s \longrightarrow \prod_s C^s \xrightarrow{1-i} \prod_s C^s \longrightarrow 0$: exact

(d) $0 \to \varprojlim_s H(C^s) \to \prod_s H(C^s) \xrightarrow{1-H(i)} \prod_s H(C^s) \to R\varprojlim_s H(C^s) \to 0$

exact

By (c), we have $\parallel \quad \circlearrowleft \quad \parallel$

$$\begin{tikzcd} H(\prod_s C^s) \arrow[rr, "H(1-i)"] & & H(\prod_s C^s) \arrow[dl, "\deg+1"] \\ & H(\varprojlim_s C^s) \arrow[ul] \arrow[u, phantom, "\text{exact}"] & \end{tikzcd}$$

Hence we have

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim_s H(C^s) \arrow[r] & \prod_s H(C^s) \arrow[r] & \prod_s H(C^s) \arrow[r] \arrow[dl, "\deg+1"] & R\varprojlim_s H(C^s) \arrow[r] & 0 \\ & & H(\varprojlim_s C^s) \arrow[u] \arrow[u, phantom, "\text{exact}"] & & & \end{tikzcd}$$

exact

$\rightsquigarrow$ Lem3.3.1 implies the exact seq.

//

§3.4 Filtered groups [Boa, §2]

Consider

G : (graded) module

$\cdots \subset F^{s+1} \subset F^s \subset \cdots \subset G$: seq of submod's

↳ [Boa]では($K=\mathbb{Z}$ なので) group の G

Def 3.4.1 [Boa, Def2.1, Prop2.2]

- $\{F^s\}$ exhausts G

  $\underset{\text{def}}{\iff} \underset{\substack{\shortparallel\\ F^{-\infty}}}{\bigcup_s F^s} = G \iff \underset{\substack{\shortparallel\\ F^{-\infty}}}{\operatorname{colim}_s F^s} = G$

- $\{F^s\}$: Hausdorff

  $\underset{\text{def}}{\iff} \underset{\substack{\shortparallel\\ F^{\infty}}}{\bigcap_s F^s} = 0 \iff \underset{\substack{\shortparallel\\ F^{\infty}}}{\lim_s F^s} = 0$

- $\{F^s\}$: complete

  $\underset{\text{def}}{\iff} \underset{\substack{\shortparallel\\ RF^{\infty}}}{R\lim_s F^s} = 0$

Rmk 3.4.2

[Boa]では.

- Def2.1 では topology の言葉で def

  ($\{F^s\}$ を使って G に topology を入れる)

- Prop2.2 で、$F^{-\infty}$, F^∞, RF^∞ を使って言いかえ

としている

↪ 面倒なので言いかえの方を def にした.

$F^{-\infty}$, F^∞, RF^∞ を詳しく調べていく.

Prop 3.4.3 [Boa, Prop2.4]

$K \subset F^\infty$: submod.

↝ G/K is filtered by $\{F^s/K\}$

Then

(a) • $\operatorname{colim}_s (F^s/K) = F^{-\infty}/K$

- $\{F^s\}$: exhausts $G \iff \{F^s/K\}$ exhausts G/K

(b) $\lim_s (F^s/K) = F^\infty/K$

(c) • $R\lim_s (F^s/K) = RF^\infty$

- $\{F^s\}$: complete $\iff \{F^s/K\}$: complete

proof

(a) $0 \to \{K\} \to \{F^s\} \to \{F^s/K\} \to 0$: exact

$\big\downarrow \operatorname{colim}_s$

$0 \to K \to F^{-\infty} \to \operatorname{colim}_s (F^s/K) \to 0$: exact

(b)(c)

Applying $\lim_s$, we have

$0 \to K \to F^\infty \to \lim_s (F^s/K)$

$\to \underset{\substack{\shortparallel\\ 0}}{R\lim_s K} \to RF^\infty \to R\lim_s (F^s/K) \to 0$: exact

$\leftarrow$ Prop3.2.5(6)

//

Rmk 3.4.4

$\lim_s (G/F^s) \neq G/F^\infty$ in general

(∵ $0 \to \{F^s\} \to \{G\} \to \{G/F^s\} \to 0$

$\big\downarrow \lim$

$0 \to F^\infty \to G \to \lim_s (G/F^s)$

$\to \underset{\substack{\nparallel\\ 0}}{RF^\infty} \to 0$: exact

in general)

◎ **Reconstitution**

$\{F^t/F^s\}_{t<s}$ ↝ recover G

Prop 3.4.5 [Boa, Prop2.5]

Assume

$\{F^s\}$: complete, Hausdorff, exhaustive

(i.e. $F^\infty = RF^\infty = 0$, $F^{-\infty} = G$)

Then

$G = \lim_s (G/F^s) = \lim_s \operatorname{colim}_t (F^t/F^s)$

proof

(1つ目の=)

$0 \to \{F^s\} \to \{G\} \to \{G/F^s\} \to 0$: exact

$\big\downarrow$

$0 \to \overset{0}{\cancel{F^\infty}} \to G \xrightarrow{\cong} \lim_s (G/F^s)$

$\to \underset{0}{\cancel{RF^\infty}} \to 0$

by assump.

(2つ目の=)

Prop3.4.3(a) + $F^{-\infty} = G$

//

comparison theorem:

Thm 3.4.6 [Boa, Thm2.6]

$f : G \longrightarrow \bar{G}$: morph of filtered modules

(i.e. $f : G \to \bar{G}$: $\mathbb{K}$-linear s.t. $\forall s$. $f(F^s) \subset \bar{F}^s$)

Assume

(1) $\{F^s\}, \{\bar{F}^s\}$: exhaustive

(i.e. $F^{-\infty} = G$, $\bar{F}^{-\infty} = \bar{G}$)

(2) f induces

$$f^\infty : F^\infty \xrightarrow{\cong} \bar{F}^\infty \text{ : isom}$$

(3) $\{F^s\}$: complete (i.e. $RF^\infty = 0$)

Rmk ($R\bar{F}^\infty = 0$ は仮定しない)

(4) $\forall s$, f induces

$$F^s/F^{s+1} \xrightarrow{\cong} \bar{F}^s/\bar{F}^{s+1} \text{ : isom}$$

Then

$f : G \xrightarrow{\cong} \bar{G}$: isom of filtered mod's

(i.e. $\cdot\ f : G \xrightarrow{\cong} \bar{G}$

$\cdot\ \forall s.\ f^s : F^s \xrightarrow{\cong} \bar{F}^s$)

(↳ $R\bar{F}^\infty = 0$)

proof

- $\forall t < \forall s\ (<\infty)$, $F^t/F^s \xrightarrow{\cong} \bar{F}^t/\bar{F}^s$

(∵ Fix t. induction on $s > t$

$$\begin{tikzcd} 0 \arrow[r] & F^s/F^{s+1} \arrow[r] \arrow[d, "\cong"', "\text{by (4)}"'] & F^t/F^{s+1} \arrow[r] \arrow[d] & F^t/F^s \arrow[r] \arrow[d, "\cong", "\text{by ind. hyp.}"] & 0 \\ 0 \arrow[r] & \bar{F}^s/\bar{F}^{s+1} \arrow[r] & \bar{F}^t/\bar{F}^{s+1} \arrow[r] & \bar{F}^t/\bar{F}^s \arrow[r] & 0 \end{tikzcd}$$
)

- $\forall s$. $G/F^s \xrightarrow{\cong} \bar{G}/\bar{F}^s$ ← (1)

(∵ Take colim_t ⇝ Prop3.4.3(a))

- $f : G \xrightarrow{\cong} \bar{G}$ (forgetting filtrations)

(∵

$$\begin{tikzcd} 0 \arrow[r] & F^\infty \arrow[r] \arrow[d, "\cong"', "(2)"'] & G \arrow[r] \arrow[d] & \lim_s G/F^s \arrow[r] \arrow[d, "\cong\ \text{上で示した}"] & \cancel{RF^\infty} = 0 \ (\leftarrow (3)) \arrow[r] \arrow[d, "\text{inj}"] & 0 \text{ : exact} \\ 0 \arrow[r] & \bar{F}^\infty \arrow[r] & \bar{G} \arrow[r] & \lim_s \bar{G}/\bar{F}^s \arrow[r] & R\bar{F}^\infty \arrow[r] & 0 \text{ : exact} \end{tikzcd}$$

↳ 5-lemma より. $G \xrightarrow{\cong} \bar{G}$: isom)

- $\forall s$. $F^s \xrightarrow{\cong} \bar{F}^s$

(∵

$$\begin{tikzcd} 0 \arrow[r] & F^s \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\cong"'] & G/F^s \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & \bar{F}^s \arrow[r] & \bar{G} \arrow[r] & \bar{G}/\bar{F}^s \arrow[r] & 0 \end{tikzcd}$$
)

//

Completion

Prop3.4.5, Thm3.4.6 are best possible results

- If $F^{-\infty} \neq G$

⇒ Consider $F^{-\infty}$ instead of G

$\{F^s\}$: filtration of $F^{-\infty}$

- If $F^\infty \neq 0$

⇒ Replace G with G/F^∞, filtered by $\{F^s/F^\infty\}$

(↘ F^t/F^s : unaffected)

- If $RF^\infty \neq 0$

↳ discussed below.

Def 3.4.7 [Boa, Def2.7]

$\hat{G} := \lim_s (G/F^s)$ the completion of G

$G \longrightarrow \hat{G}$: the completion hom

(induced by $G \xrightarrow{\text{proj}} G/F^s$)

↑ filter $\hat{G}$ by

$\hat{F}^t := \lim_s F^t/F^s \subset \hat{G}$

↑ $\lim_s$: left exact

Prop3.4.8 [Boa, Prop2.8]

(a) $\{\hat{F}^t\}$: complete Hausdorff

(i.e. $\hat{F}^\infty = R\hat{F}^\infty = 0$)

(b) $G \longrightarrow \hat{G}$ induces

$\cdot\ F^t/F^s \xrightarrow{\cong} \hat{F}^t/\hat{F}^s \quad (\forall t < \forall s)$

$\cdot\ G/F^s \xrightarrow{\cong} \hat{G}/\hat{F}^s \quad (\forall s)$

(c) $G/F^{-\infty} \xrightarrow{\cong} \hat{G}/\hat{F}^{-\infty}$

↘ $\{\hat{F}^t\}$ exhausts $\hat{G}$ ⟺ $\{F^t\}$ exhausts G

proof

(b) $0 \to F^t/F^s \to G/F^s \to G/F^t \to 0$ — (*)

↓ $\lim_s$

$0 \to \hat{F}^t \to \hat{G} \to G/F^t \to 0$: exact — (**)

(∵ $R\lim_s (F^t/F^s) = 0$ by Prop3.2.5(b))

↳ $\hat{G}/\hat{F}^t \xrightarrow{\cong} G/F^t$ ($G \to \hat{G}$ induces the inverse)

- $F^t/F^s \xrightarrow{\cong} \hat{F}^t/\hat{F}^s$ は. (*) とその $\hat{G}$ 版に5-lemma

(a) Apply $\lim_t$ to (**)

↳ $0 \to \hat{F}^\infty \to \hat{G} \xrightarrow{\cong} \lim_t (G/F^t) \to R\hat{F}^\infty \to 0$: exact (by def of $\hat{G}$)

↘ $\hat{F}^\infty = R\hat{F}^\infty = 0$

(c) Apply colim_t to (**)

//

Ex 3.4.9

$G := \mathbb{K}[x]$

$$F^s := \begin{cases} \mathbb{K}[x] & (s \leq 0) \\ x^s \mathbb{K}[x] & (s \geq 0) \end{cases}$$

↖ ideal gen'd by x^s (in $\mathbb{K}[x]$)

Then

- $\hat{G} \cong \mathbb{K}[[x]]$
- $\hat{F}^t = x^t \mathbb{K}[[x]]$

By Prop 3.4.8

$$R\hat{F}^\infty = 0 \quad \text{——} \quad (*)$$

$(*)$の"気持ち"をつかむために、直接証明してみる

proof of $(*)$

Enough to show:

$$1-i : \prod_{t \geq 0} x^t \mathbb{K}[[x]] \longrightarrow \prod_{t \geq 0} x^t \mathbb{K}[[x]] \ : \ \text{surj}$$

$$\{x^t f_t(x)\}_t \longmapsto \{x^t f_t(x) - x^{t+1} f_{t+1}(x)\}_t$$

Take $\forall \{x^t g_t(x)\}_t \in \prod_{t \geq 0} x^t \mathbb{K}[[x]]$

↳ We need to find $\{x^t f_t(x)\}_t$ s.t.

$\forall t,\ x^t f_t(x) - x^{t+1} f_{t+1}(x) = x^t g_t(x)$

i.e. $f_t(x) = g_t(x) + x f_{t+1}(x)$

Define $f_t(x)$ by

$$f_t(x) := \sum_{n \geq 0} x^n g_{t+n}(x) \in \mathbb{K}[[x]]$$

formal power series だから well-defd.

($\mathbb{K}[x]$ だと無理)

Then we have

$$x^t f_t(x) - x^{t+1} f_{t+1}(x)$$

$$= x^t \left(\sum_{n \geq 0} x^n g_{t+n}(x) - x \sum_{n \geq 0} x^n g_{t+n+1}(x) \right)$$

$$= x^t g_t(x)$$

//

↪「completion したおかげで無限和がとれる」ことが、ちゃんと $R\hat{F}^\infty = 0$ の鍵になっているとわかる.

$\mathbb{K}[x]$ を $\mathbb{Z}$ におきかえても同様の例ができる:

Ex 3.4.10

$n \geq 2$

$G := \mathbb{Z}$

$$F^s := \begin{cases} \mathbb{Z} & (s \leq 0) \\ n^s \mathbb{Z} & (s \geq 0) \end{cases}$$

Then

- $\hat{G} = \hat{\mathbb{Z}}_n \left(:= \varprojlim_s \mathbb{Z}/n^s \right)$
- $\hat{F}^t = n^t \hat{\mathbb{Z}}_n$

§3.5 Image subsequence and Mittag-Leffler exact sequence [Boa, §3]

$\{A^s\}_s$: sequence

Recall

$\mathrm{Im}^r A^s := \mathrm{Im}(i^r : A^{s+r} \to A^s)$

Def3.5.1 [Boa, Def 3.1]

$Q^s := \varprojlim_r \mathrm{Im}^r A^s = \bigcap \mathrm{Im}^r A^s \subset A^s$

$RQ^s := R\varprojlim_r \mathrm{Im}^r A^s$

Rmk3.5.2

- RQ^s is introduced by the "policy" (see §3.2)
- RQ^∞ is ambiguous

  $R\varprojlim_s Q^s$, $\varprojlim_s RQ^s$ ← write in this way

Thm3.5.3 [Boa, Thm3.4]

(a) $Q^s \hookrightarrow A^s$ induces

$$\varprojlim_s Q^s \xrightarrow{\cong} \varprojlim_s A^s$$

← mapについては Cor3.5.15を参照

(b) $0 \to R\varprojlim_s Q^s \to RA^\infty \to \varprojlim_s RQ^s \to 0$: exact

Mittag-Leffler exact seq.

(c)
- $\forall s$, $RQ^{s+1} \twoheadrightarrow RQ^s$: surj.
- $R\varprojlim_s RQ^s = 0$ (← induced by $\mathrm{Im}^r A^{s+1} \xrightarrow{i} \mathrm{Im}^r A^s$)

proof of (a) and (c)

(a) We have

$$\varprojlim_s A^s \longrightarrow Q^t \quad \text{for } \forall t.$$

$$(a^s) \longmapsto a^t \quad (∵ \cdots \mapsto a^{t+1} \mapsto a^t)$$

universality of $\varprojlim_t$

$$\varprojlim_s A^s \longrightarrow \varprojlim_t Q^t$$

This gives the inverse.

(c) $\mathrm{Im}^r A^{s+1} \overset{i}{\twoheadrightarrow} \mathrm{Im}^{r+1} A^s$: surj

$R\varprojlim_r$: right exact

$$\begin{tikzcd} R\varprojlim_r \mathrm{Im}^r A^{s+1} \arrow[r, two heads, "i"] \arrow[dr, "i", "\text{surj}"'] \arrow[dr, phantom, shift left=4, "\circlearrowleft"] & R\varprojlim_r \mathrm{Im}^{r+1} A^s \arrow[d, "\cong"] \\ & R\varprojlim_r \mathrm{Im}^r A^s \end{tikzcd}$$

: surj

(∵ Prop3.2.14)

ind : $\mathrm{Im}^{r+1}A^s \to \mathrm{Im}^r A^s$ is the str. map

また、Prop3.2.5(b) より

$R\varprojlim RQ^s = 0$

//

Rmk3.5.4

- (b) の proof は少し大変なので後まわし.
- (b) について:

$$\begin{tikzcd} 0 \arrow[r] & R\varprojlim_s Q^s \arrow[r] \arrow[d, equal] & RA^\infty \arrow[r] & \varprojlim_s RQ^s \arrow[r] \arrow[d, equal] & 0 \\ & R\varprojlim_s \varprojlim_r \mathrm{Im}^r A^s & & \varprojlim_s R\varprojlim_r \mathrm{Im}^r A^s & \end{tikzcd}$$

Cor3.5.5 [Boa, Cor3.6]

$RA^\infty = 0 \Rightarrow \forall s,\ RQ^s = 0$

proof

$$\left.\begin{array}{l} \text{Thm3.5.3(b)} \rightsquigarrow \varprojlim_s RQ^s = 0 \\ \text{(c)} \rightsquigarrow RQ^{s+1} \underset{\text{surj}}{\twoheadrightarrow} RQ^s \end{array}\right) \xrightarrow{\text{Cor3.2.6}} \forall s.\ RQ^s = 0$$

//

Cor3.5.6 (Mittag-Leffler condition)

Assume

$\forall s,\ \exists r_0(s)$ s.t. $\forall r \geq r_0(s)$, $\mathrm{Im}^r A^s = \mathrm{Im}^{r_0(s)} A^s$

Then

$RA^\infty = 0$

proof

By assump,
- $\forall s$, $RQ^s = 0$ ($\rightsquigarrow \varprojlim_s RQ^s = 0$) ①
- $\forall r \geq r_0(s)$, $Q^s = \mathrm{Im}^r A^s$

(∵ $\mathrm{Im}^r A^s$ is constant for large r ($\geq r_0(s)$))

$\forall s$. $Q^{s+1} \twoheadrightarrow Q^s$: surj.

(∵ $r \geq \max\{r_0(s)-1, r_0(s+1)\}$ なるrをとると.

$$\begin{array}{ccc} Q^{s+1} & = & \mathrm{Im}^r A^{s+1} \\ & & \downarrow \text{surj} \\ Q^s & = & \mathrm{Im}^{r+1} A^s \end{array}$$

)

よって Prop3.2.5(b) より.

$R\varprojlim_s Q^s = 0$ — ②

①②と Thm3.5.3(b) より.

$RA^\infty = 0$

//

Rmk3.5.7

Cor3.5.6 を示すだけならもう少し楽な方法もある

(e.g. [Wei, Prop3.5.7])

Thm 3.5.3 (b) を示すために準備.

Lem 3.5.8 [Boa, in the proof of Thm 3.4]

Assume

$\forall s$, $\{\mathrm{Im}^r A^s\}_r$: complete Hausdorff

(i.e. $Q^s = RQ^s = 0$)

Then

$$A^\infty = RA^\infty = 0$$

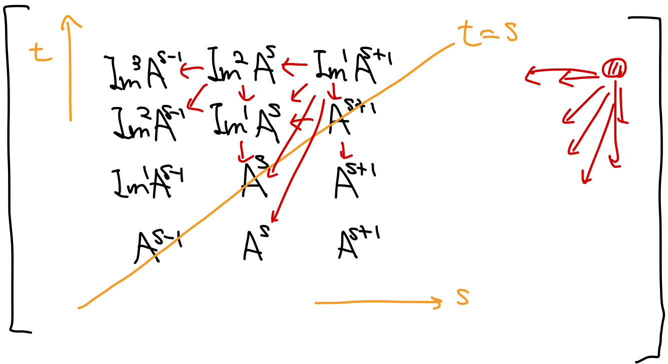
proof

$A^\infty = 0$ は明らか. (∵ $A^\infty \overset{(a)}{=} Q^\infty = 0$)

Define

$$I^{st} := \mathrm{Im}\left(A^{\max\{s,t\}} \longrightarrow A^s\right) = \begin{cases} \mathrm{Im}^{t-s} A^s & (t \geq s) \\ A^s & (t \leq s) \end{cases}$$

with $I^{st} \longrightarrow I^{uv}$ for $s \geq u$ and $t \geq v$



For any t, we have

$$\lim_s I^{st} = 0, \quad R\lim_s I^{st} = RA^\infty$$

(∵ Fix t.
$\forall s \geq t$. $I^{st} = A^s$
⇝ $\lim_s I^{st} = A^\infty = 0$, $R\lim_s I^{st} = RA^\infty$)

Hence, by the definition of $\lim_s$ and $R\lim_s$, we have

$$0 \to P^t \xrightarrow{\boxed{1-i}} P^t \longrightarrow RA^\infty \longrightarrow 0 \text{ : exact}$$

$$\left(\underline{\text{where}} \quad P^t := \prod_s I^{st}\right) \quad \text{―― (*)}$$

Here we have

$$P^\infty = RP^\infty = 0$$

(∵ $\begin{cases} \lim_t I^{st} = \lim_t \mathrm{Im}^{t-s} A^s = Q^s = 0 \\ R\lim_t I^{st} = R \text{ ―――― } = RQ^s = 0 \end{cases}$ ← assump.

$\xrightarrow{\text{Prop 3.2.10}}$ $P^\infty = RP^\infty = 0$)

Apply $\lim_t$ to (*)

Rmk
この map が id ではないので, $P^\infty = RP^\infty = 0$ が必要だった

$$⇝ \quad 0 \to P^\infty \xrightarrow{1-i} \overset{0}{\cancel{P^\infty}} \longrightarrow RA^\infty$$

$$\to \overset{0}{\cancel{RP^\infty}} \longrightarrow RP^\infty \longrightarrow 0 \quad \text{: exact}$$

$$⇝ RA^\infty = 0$$

//

Def 3.5.9 [Boa, in the proof of Thm 3.4]

Define a seq $\{\hat{A}^s\}_s$ by

- $\hat{A}^s :=$ (completion of A^s w.r.t $\{\mathrm{Im}^t A^s\}_t$)
  $= \lim_t A^s/\mathrm{Im}^t A^s$
- $i : \hat{A}^{s+1} \longrightarrow \hat{A}^s$ is defined by
  $i : A^{s+1}/\mathrm{Im}^t A^{s+1} \longrightarrow A^s/\mathrm{Im}^t A^s$

We have two filtrations on $\hat{A}^s$

- $\mathrm{Im}^r \hat{A}^s = \mathrm{Im}(\hat{A}^{s+r} \longrightarrow \hat{A}^s)$
- $F^r \hat{A}^s := \lim_t \left(\mathrm{Im}^r A^s / \mathrm{Im}^t A^s\right)$: complete Hausdorff (Prop 3.4.8 (a))
  (defined after Def 3.4.7)

Lem 3.5.10 [Boa, in the proof of Thm 3.4]

(1) $\forall r, s$, $\mathrm{Im}^r \hat{A}^s = F^r \hat{A}^s$ (as submod's of $\hat{A}^s$)

(2) $\hat{A}^\infty = R\hat{A}^\infty = 0$

proof (1)

(⊂) $\mathrm{Im}^r \hat{A}^s = \mathrm{Im}(\hat{A}^{s+r} \longrightarrow \hat{A}^s)$

$$= \mathrm{Im}\left(\lim_t \left(A^{s+r}/\mathrm{Im}^t A^{s+r}\right) \xrightarrow{i^r} \lim_t \left(A^s/\mathrm{Im}^{t+r} A^s\right)\right)$$

∩

これ i^r

$$F^r \hat{A}^s = \lim_t \left(\mathrm{Im}^r A^s / \mathrm{Im}^{t+r} A^s\right)$$

(⊃) Enough to show :

$$\lim_t i^r : \lim_t \left(A^{s+r}/\mathrm{Im}^t A^{s+r}\right) \longrightarrow \lim_t \left(\mathrm{Im}^r A^s/\mathrm{Im}^{t+r} A^s\right) \text{ : surj}$$

Before taking $\lim_t$, we have :

$$\underset{\substack{\parallel \\ \mathrm{Ker}(i^r : A^{s+r} \to A^s)}}{\mathrm{Ker}^r A^{s+r}} \xrightarrow{\varphi} A^{s+r}/\mathrm{Im}^t A^{s+r} \xrightarrow{i^r} \mathrm{Im}^r A^s/\mathrm{Im}^{t+r} A^s \to 0 \text{ : exact}$$

(∵ $\underline{i^r : \text{surj}}$ 明らか.
$\underline{i^r \circ \varphi = 0}$ 明らか.
$\underline{\mathrm{Ker}\, i^r \subset \mathrm{Im}\, \varphi}$ $[a] \in \mathrm{Ker}\, i^r$ とすると, $i^r(a) \in \mathrm{Im}^{t+r} A^s$
⇝ $\exists b.$ $i^r(a) = i^{t+r}(b)$
⇝ $\begin{cases} \cdot\ a - i^t(b) \in \mathrm{Ker}^r A^{s+r} \\ \cdot\ \varphi(a - i^t(b)) = [a] - \overset{0}{\cancel{[i^t(b)]}} = [a] \end{cases}$)

Since $\mathrm{Ker}^r A^{s+r}$: const. on t, we have

$$R\lim_t \mathrm{Ker}^r A^{s+r} = 0$$

Hence by Cor 3.2.4,

$$\lim_t i^r : \text{surj.}$$

(2) As noted above,

$$\{F^r \hat{A}^s\}_r \text{ : complete Hausdorff}$$

(1) ― $\parallel$

$\{\mathrm{Im}^r \hat{A}^s\}_r$ ⇝ Apply Lem 3.5.8

//

Thm3.5.3 (再掲)

(6) $0 \to \mathrm{R}\lim_s Q^s \to RA^\infty \to \lim_s RQ^s \to 0$: exact

proof

$0 \to \mathrm{Im}^r A^s \to A^s \to A^s/\mathrm{Im}^r A^s \to 0$: exact

$\rightsquigarrow \lim_r$

$0 \to Q^s \to A^s \to \hat{A}^s \to RQ^s \to 0$: exact

Define

$J^s := A^s/Q^s$

Then we have :

$$\begin{cases} 0 \to Q^s \to A^s \to J^s \to 0 : \text{exact} & \text{— ①} \\ 0 \to J^s \to \hat{A}^s \to RQ^s \to 0 : \text{exact} & \text{— ②} \end{cases}$$

(😀 Lem3.5.10(2))

$\lim_s ② \rightsquigarrow 0 \to J^\infty \to \overset{0}{\cancel{\hat{A}^\infty}} \to \lim_s RQ^s$

$\xrightarrow{\delta} RJ^\infty \to \underset{0}{\cancel{R\hat{A}^\infty}} \to \mathrm{R}\lim_s RQ^s \to 0$: exact

$\rightsquigarrow J^\infty = 0,\ RJ^\infty \cong \lim_s RQ^s$

$\lim_s ① \rightsquigarrow 0 \to Q^\infty \to A^\infty \to \overset{0}{\cancel{J^\infty}}$

$\to \mathrm{R}\lim_s Q^s \to RA^\infty \to \underset{\lim_s RQ^s}{\underset{\parallel}{\boxed{RJ^\infty}}} \to 0$: exact

//

Rmk 3.5.11

[Boa] には

「The result can be considered an application of the spectral sequence of the double limit system $\mathrm{Im}^r A^s$」

と書いてあるけど…?

何これ?

[Boa] には書いていないけど、
ML exact seq に現れる map をちゃんと調べておく。

Def 3.5.12

Define maps

$$\begin{tikzcd} \mathrm{R}\lim_s Q^s \arrow[r, "RL"] & RA^\infty \arrow[r, "\varphi"] \arrow[dr, "R\pi"'] & \lim_s RQ^s \arrow[d, "\delta", "\cong"'] \\ & & \mathrm{R}\lim(A^s/Q^s) \end{tikzcd}$$

by

- $Q^s \xrightarrow{L} A^s \xrightarrow{\pi} A^s/Q^s$

  $\rightsquigarrow \mathrm{R}\lim_s$

  $\mathrm{R}\lim_s Q^s \xrightarrow{RL} \mathrm{R}\lim_s A^s \xrightarrow{R\pi} \mathrm{R}\lim_s (A^s/Q^s)$

- δ is the connecting hom. defined in the proof of Thm3.5.3(6)

- $A^{s+r} \longrightarrow \mathrm{Im}^r A^s$

$$\begin{tikzcd} \mathrm{R}\lim_r A^{s+r} \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{R}\lim_r \mathrm{Im}^r A^s \arrow[d, phantom, "\parallel"] \\ RA^\infty \arrow[r] & RQ^s \end{tikzcd}$$

(with $\rightsquigarrow \mathrm{R}\lim_r$ before)

$$\begin{tikzcd} \lim_s RA^\infty \arrow[r] \arrow[d, phantom, "\parallel"] & \lim_s RQ^s \\ RA^\infty \arrow[ur, "\varphi"'] & \end{tikzcd}$$

(with $\rightsquigarrow \lim_s$ before)

Prop 3.5.13

$0 \to \mathrm{R}\lim_s Q^s \xrightarrow{RL} RA^\infty \xrightarrow{\delta^{-1} \circ R\pi} \lim_s RQ^s \to 0$

exact

proof 左にある Thm 3.5.3(6) の proof は、
実際にはこれを証明している。

//

Prop 3.5.14

$$\begin{tikzcd} RA^\infty \arrow[r, "\varphi"] \arrow[dr, "R\pi"'] \arrow[dr, phantom, "(-1)", shift left=3] & \lim_s RQ^s \arrow[d, "\delta", "\cong"'] \\ & \mathrm{R}\lim(A^s/Q^s) \end{tikzcd}$$

commutative with the sign (-1)

(i.e. $\delta \circ \varphi = -R\pi$)

Cor 3.5.15

$0 \to \mathrm{R}\lim_s Q^s \xrightarrow{RL} RA^\infty \xrightarrow{\varphi} \lim_s RQ^s \to 0$: exact

proof Prop3.5.13 + Prop 3.5.14 //

proof of Prop3.5.14

（連結準同型も含めて全ての map を具体的に計算するという. 泥臭い証明とする）

$$R\pi : R\varprojlim_s A^s \longrightarrow R\varprojlim_s (A^s/Q^s)$$
$$[\underbrace{\{a^s\}_s}] \longmapsto [\underbrace{\{[a^s]\}_s}]$$

（$\prod_s A^s$ ⫛ ↑ equiv class　$\prod_s A^s/Q^s$ ⫛

$R\varprojlim_s A^s = \mathrm{Coker}(1-i : \prod_s A^s \to \prod_s A^s)$）

$$\varphi : R\varprojlim_s A^s \longrightarrow \varprojlim_s R\varprojlim_r \mathrm{Im}^r A^s$$
$$[\{a^s\}_s] \longmapsto \{[\{i^r(a^{s+r})\}_r]\}_s$$

（∵ $A^{s+r} \longrightarrow \mathrm{Im}^r A^s$

$a^{s+r} \longmapsto i^r(a^{s+r})$

$\xrightarrow{R\varprojlim_r}$ $R\varprojlim_r A^{s+r} \longrightarrow R\varprojlim_r \mathrm{Im}^r A^s$

$\cong$　$[\{a^{s+r}\}_r] \longmapsto [\{i^r(a^{s+r})\}_r]$

$R\varprojlim_r A^r \ni [\{a^r\}_r]$ ↕

$\xrightarrow{\varprojlim_s}$ $R\varprojlim_r A^r \longrightarrow \varprojlim_s R\varprojlim_r \mathrm{Im}^r A^s$

$[\{a^r\}_r] \longmapsto \{[\{i^r(a^{s+r})\}_r]\}_s$ ）

Recall the construction of δ:

$0 \to \mathrm{Im}^r A^s \longrightarrow A^s \longrightarrow A^s/\mathrm{Im}^r A^s \to 0$: exact

$\downarrow R\varprojlim_r$

$0 \to Q^s \longrightarrow A^s \to \varprojlim_r A^s/\mathrm{Im}^r A^s \xrightarrow{\partial} R\varprojlim_r \mathrm{Im}^r A^s \to 0$
exact

$\downarrow$

$0 \to A^s/Q^s \to \varprojlim_r A^s/\mathrm{Im}^r A^s \xrightarrow{\partial} R\varprojlim_r \mathrm{Im}^r A^s \to 0$
exact

$\downarrow \varprojlim_s$

$0 \to \varprojlim_s A^s/Q^s \to \varprojlim_s \varprojlim_r A^s/\mathrm{Im}^r A^s \xrightarrow{\varprojlim_s \partial} \varprojlim_s R\varprojlim_r \mathrm{Im}^r A^s$

$\xrightarrow{\delta} R\varprojlim_s A^s/Q^s \to R\varprojlim_s \varprojlim_r A^s/\mathrm{Im}^r A^s \longrightarrow \cdots$: exact

First, we compute ∂

$$\partial : \varprojlim_r A^s/\mathrm{Im}^r A^s \longrightarrow R\varprojlim_r \mathrm{Im}^r A^s$$
$$\{[a_r^s]\}_r \longmapsto [\{a_r^s - a_{r+1}^s\}_r]$$

（∵

$\{a_r^s\} \longmapsto \{[a_r^s]\}_r \in \mathrm{Ker}(1-\sigma)$

$$\begin{tikzcd}
0 \arrow[r] & \prod_r \mathrm{Im}^r A^s \arrow[r] \arrow[d, "1-\sigma"] & \prod_r A^s \arrow[r] \arrow[d, "1-\sigma"] & \prod_r A^s/\mathrm{Im}^r A^s \arrow[r] \arrow[d, "1-\sigma"] & 0 \\
0 \arrow[r] & \prod_r \mathrm{Im}^r A^s \arrow[r] & \prod_r A^s \arrow[r] & \prod_r A^s/\mathrm{Im}^r A^s \arrow[r] & 0
\end{tikzcd}$$

$\{a_r^s - a_{r+1}^s\}_r \longmapsto \{a_r^s - a_{r+1}^s\} \longmapsto 0$

（$\sigma : \prod_r A^s \longrightarrow \prod_r A^s$

$\{a_r^s\}_r \longmapsto \{a_{r+1}^s\}$ ）← sをfixしているので iではなく shift

）

Now we compute $\delta \circ \varphi$

$$\delta \circ \varphi(\underbrace{[\{a^s\}_s]}) = \varphi(\{[\{i^r(a^{s+r})\}_r]\}_s)$$

arbitrary element in RA^∞

$\{\{[a_r^s]\}_r\}_s \overset{\prod_s \partial}{\underset{①}{\longmapsto}} \{[\{i^r(a^{s+r})\}_r]\}_s$

$$\begin{tikzcd}
0 \arrow[r] & \prod_s A^s/Q^s \arrow[r] \arrow[d, "1-i"] & \prod_s \varprojlim_r A^s/\mathrm{Im}^r A^s \arrow[r, "\prod_s \partial"] \arrow[d, "1-i"] & \prod_s R\varprojlim_r \mathrm{Im}^r A^s \arrow[r] \arrow[d, "1-i"] & 0 \\
0 \arrow[r] & \prod_s A^s/Q^s \arrow[r] & \prod_s \varprojlim_r A^s/\mathrm{Im}^r A^s \arrow[r] & \prod_s R\varprojlim_r \mathrm{Im}^r A^s \arrow[r] & 0
\end{tikzcd}$$

sを動かしているので こっちは i →

$\{[-a^s]\}_s \longmapsto \{\{[-a^s]\}_r\}_s \overset{②}{=} \{\{[a_r^s - i(a_r^{s+1})]\}_r\}_s$

$\{\{[a_r^s]\}_r - i\{[a_r^s]\}_r\}_s$ ‖ $\{\{[a_r^s - i(a_r^{s+1})]\}_r\}_s$

（where

$$a_r^s := -\sum_{0 \le t < r} i^t(a^{s+t})$$

↳ ① $a_r^s - a_{r+1}^s = i^r(a^{s+r})$

② $a_r^s - i(a_r^{s+1}) = -\sum_{0 \le t < r} i^t(a^{s+t}) + i \sum_{0 \le t < r} i^t(a^{s+t+1})$

$$= -a^s + \underbrace{i^r(a^{s+r})}_{\in \mathrm{Im}^r A^s}$$
）

Hence

$$\delta \circ \varphi([\{a^s\}_s]) = \{[-a^s]\}_s = -R\pi([\{a^s\}_s])$$

//

↑ もっとマシな証明ないの？とは思うけど，

連結準同型の処理がうまくできない.

Q^sとA^∞を比較する.

Def 3.5.16

$$\mathrm{Im}^{\sigma} A^s := \mathrm{Im}(\varepsilon^s : A^\infty \to A^s)$$

↑ このnotationの由来については§3.6を参照.

Lem 3.5.17

(1) $Q^s = \{a^s \in A^s \mid \forall r \geq 0, \exists a^{s+r}_{(r)} \mapsto \exists a^{s+r-1}_{(r)} \mapsto \cdots \mapsto \exists a^{s+1}_{(r)} \mapsto a^s\}$

← 任意の長さの有限列がとれる

$\mathrm{Im}^{\sigma} A^s = \{a^s \in A^s \mid \cdots \mapsto \exists a^{s+2} \mapsto \exists a^{s+1} \mapsto a^s\}$

← 無限列が一斉にとれる

($\mathrm{Im}^{\sigma} A^s$については, 自動的に次が成立:
$\forall r \geq 0, \ a^{s+r} \in \mathrm{Im}^{\sigma} A^{s+r}$)

(2) $\mathrm{Im}^{\sigma} A^s \subset Q^s \ (\subset A^s)$

proof 明らか. //

Prop 3.5.18

$f : \{A^s\}_s \to \{\bar{A}^s\}_s$: morph of seq's

Fix s_0

Assume

$\forall s \geq s_0, \ f : Q^s \xrightarrow{\cong} \bar{Q}^s$: isom

Then

$\forall s \geq s_0, \ f : \mathrm{Im}^{\sigma} A^s \xrightarrow{\cong} \mathrm{Im}^{\sigma} \bar{A}^s$: isom

proof

$$\begin{tikzcd} Q^s \arrow[r, "\cong"] & \bar{Q}^s \\ \mathrm{Im}^{\sigma} A^s \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Im}^{\sigma} \bar{A}^s \arrow[u, hook] \end{tikzcd}$$

inj $\mathrm{Im}^{\sigma} A^s \hookrightarrow Q^s \xrightarrow{\cong} \bar{Q}^s$: inj より ok

surj $\forall s$ と $\forall \bar{a}^s \in \mathrm{Im}^{\sigma} \bar{A}^s$ を fix する

($\rightsquigarrow \exists a^s \in \mathrm{Im}^{\sigma} A^s$ s.t. $f(a^s) = \bar{a}^s$ を示したい)

Lem 3.5.17 (1) より

$\cdots \mapsto \exists \bar{a}^{s+2} \mapsto \exists \bar{a}^{s+1} \mapsto \bar{a}^s$

with $\forall r$. $\bar{a}^{s+r} \in \mathrm{Im}^{\sigma} \bar{A}^{s+r} \subset \bar{Q}^{s+r}$

$f : Q^{s+r} \xrightarrow{\cong} \bar{Q}^{s+r}$: isom なので.

$a^{s+r} := f^{-1}(\bar{a}^{s+r}) \in Q^{s+r}$

と def する. このとき $\forall r \geq 1$ に対し

$f(i(a^{s+r})) = i f(a^{s+r}) = i(\bar{a}^{s+r}) = \bar{a}^{s+r-1} = f(a^{s+r-1})$

$\xrightarrow[f:\text{inj}]{} i(a^{s+r}) = a^{s+r-1}$

よって $a^s \in \mathrm{Im}^{\sigma} A^s$ (and $f(a^s) = \bar{a}^s$) //

Cor 3.5.19

$f : \{A^s\}_s \to \{\bar{A}^s\}_s$: morph of seq's

Assume

$\forall s, \ f : Q^s \xrightarrow{\cong} \bar{Q}^s$: isom

Then

$\forall s, \ f : \mathrm{Im}^{\sigma} A^s \xrightarrow{\cong} \mathrm{Im}^{\sigma} \bar{A}^s$: isom

proof Prop 3.5.18 より直ちに従う. //

$\mathrm{Im}^{\sigma} A^s \subsetneq Q^s$ となる例を与えておく.

Ex 3.5.20

(1)

$$\begin{tikzcd} \cdots \arrow[r] & A^3 \arrow[r] \arrow[d, equal] & A^2 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, equal] & A^0 \arrow[r] \arrow[d, equal] & A^{-1} \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \mathbb{K}[x] \arrow[r, "x\times"] & \mathbb{K}[x] \arrow[r, "x\times"] & \mathbb{K}[x] \arrow[r] & \mathbb{K} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

$f \longmapsto f(1)$

このとき

- $Q^0 = \mathbb{K}$
- $A^\infty = 0, \ \mathrm{Im}^{\sigma} A^0 = 0$

$\rightsquigarrow \mathrm{Im}^{\sigma} A^0 \subsetneq Q^0$

- $RA^\infty = \mathbb{K}[[x]]/\mathbb{K}[x]$ (quotient as $\mathbb{K}$-mod's)

(∵ Ex 3.2.9)

(2)

$$\begin{tikzcd} \cdots \arrow[r] & A^3 \arrow[r] \arrow[d, equal] & A^2 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, equal] & A^0 \arrow[r] \arrow[d, equal] & A^{-1} \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \mathbb{K}[[x]] \arrow[r, "x\times"] & \mathbb{K}[[x]] \arrow[r, "x\times"] & \mathbb{K}[[x]] \arrow[r, two heads, "\mathrm{proj}"] & \mathbb{K}[[x]]/\mathbb{K}[x] \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

このとき

- $Q^0 = \mathbb{K}[[x]]/\mathbb{K}[x]$
- $A^\infty = 0, \ \mathrm{Im}^{\sigma} A^0 = 0$

$\rightsquigarrow \mathrm{Im}^{\sigma} A^0 \subsetneq Q^0$

- $RA^\infty = 0$

(∵ $\{A^s\}_{s \geq 1}$ は次と同一視できる:
$\cdots \to x^2\mathbb{K}[[x]] \to x\mathbb{K}[[x]] \to \mathbb{K}[[x]] \to 0 \to \cdots$
よって Ex 3.4.9, Prop 3.4.8 より ok.)

Ex 3.5.21

Ex 3.5.20 の $\mathbb{K}[x]$ を $\mathbb{Z}$ に置き換えても同様の例が得られる.

(1) $\cdots \to \mathbb{Z} \xrightarrow{n\times} \mathbb{Z} \xrightarrow{n\times} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n-1 \to 0 \to \cdots$ $(n \geq 3)$

$\rightsquigarrow Q^0 = \mathbb{Z}/n-1, \ A^\infty = 0, \ \mathrm{Im}^{\sigma} A^0 = 0,$

$RA^\infty = \hat{\mathbb{Z}}_n/\mathbb{Z}$

(2) $\cdots \to \hat{\mathbb{Z}}_n \xrightarrow{n\times} \hat{\mathbb{Z}}_n \xrightarrow{n\times} \hat{\mathbb{Z}}_n \twoheadrightarrow \hat{\mathbb{Z}}_n/\mathbb{Z} \to 0 \to \cdots$ $(n \geq 2)$

$\rightsquigarrow Q^0 = \hat{\mathbb{Z}}_n/\mathbb{Z}, \ A^\infty = RA^\infty = 0, \ \mathrm{Im}^{\sigma} A^0 = 0$

§3.6 Interaction between limits and colimits

[Boa, §8]

seqの limit と colimit が両方現れる場合には、その2つの interaction が問題になる。

(§4.4 Whole-plane spectral sequences でしか使われない)

Recall $(s\in\mathbb{Z})$

- $\mathrm{Im}^r A^s := \mathrm{Im}(A^{s+r} \longrightarrow A^s)$ $\quad (\overset{0\le r<\omega}{r\in\mathbb{N}})$
  $\mathrm{Im}^\omega A^s := \lim_r \mathrm{Im}^r A^s = \bigcap_r \mathrm{Im}^r A^s$
- $Q^s := \lim_r \mathrm{Im}^r A^s = \mathrm{Im}^\omega A^s$
  $RQ^s := R\lim_r \mathrm{Im}^r A^s$
- $K_n A^s := \mathrm{Ker}(A^s \longrightarrow A^{s-n})$ $\quad (0\le n<\infty)$

Def 3.6.1

$$K_\infty A^s := \mathrm{colim}_n K_n A^s = \bigcup_n K_n A^s = \mathrm{Ker}(\eta^s : A^s \longrightarrow A^{-\infty})$$

(∵ colim : exact)

Def 3.6.2

For $0\le r\le \omega$, $0\le n\le\infty$, define

$$K_n\mathrm{Im}^r A^s := K_n A^s \cap \mathrm{Im}^r A^s$$

$\lim_r$ と colim_n は交換する:

Lem 3.6.3

As submodules of A^s, the followings hold

(1) $0\le {}^\forall n\le\infty$
$$\lim_r K_n\mathrm{Im}^r A^s = K_n\mathrm{Im}^\omega A^s$$

(2) $0\le {}^\forall r\le\omega$
$$\mathrm{colim}_n K_n\mathrm{Im}^r A^s = K_\infty\mathrm{Im}^r A^s$$

(3) $$\lim_r \mathrm{colim}_n K_n\mathrm{Im}^r A^s = \mathrm{colim}_n \lim_r K_n\mathrm{Im}^r A^s = K_\infty\mathrm{Im}^\omega A^s$$

proof

(1) $$\lim_r K_n\mathrm{Im}^r A^s = \bigcap_r (K_nA^s\cap\mathrm{Im}^r A^s) = K_nA^s\cap\Big(\bigcap_r \mathrm{Im}^r A^s\Big) = K_nA^s\cap\mathrm{Im}^\omega A^s = K_n\mathrm{Im}^\omega A^s$$

(2) $$\mathrm{colim}_n K_n\mathrm{Im}^r A^s = \bigcup_n (K_nA^s\cap\mathrm{Im}^r A^s) = \Big(\bigcup_n K_nA^s\Big)\cap\mathrm{Im}^r A^s = K_\infty A^s\cap\mathrm{Im}^r A^s = K_\infty\mathrm{Im}^r A^s$$

(3) $$\lim_r \mathrm{colim}_n K_n\mathrm{Im}^r A^s$$
$$= \lim_r K_\infty\mathrm{Im}^r A^s \quad (∵ (2)\ r<\omega)$$
$$= K_\infty\mathrm{Im}^\omega A^s \quad (∵ (1)\ n=\infty)$$

$$\mathrm{colim}_n \lim_r K_n\mathrm{Im}^r A^s$$
$$= \mathrm{colim}_n K_n\mathrm{Im}^\omega A^s \quad (∵ (1)\ n<\infty)$$
$$= K_\infty\mathrm{Im}^\omega A^s \quad (∵ (2)\ r=\omega)$$

//

しかし、$R\lim_r$ と colim_n は交換しない:

Def 3.6.4

$$W := \mathrm{colim}_s R\lim_r K_\infty\mathrm{Im}^r A^s$$

Lem 3.6.5 [Boa, Lem 8.5]

For ${}^\forall s\in\mathbb{Z}$, we have:

(1) $$0\to \mathrm{colim}_n R\lim_r K_n\mathrm{Im}^r A^s \to R\lim_r \mathrm{colim}_n K_n\mathrm{Im}^r A^s \to W \to 0 \ : \text{exact}$$

(2) $$W \cong \mathrm{colim}_n R\lim_r \frac{K_\infty\mathrm{Im}^r A^s}{K_n\mathrm{Im}^r A^s}$$
(independent of s)

Lem を準備する:

Lem 3.6.6

$0\le{}^\forall r\le\omega$,
$$\mathrm{colim}_s K_\infty\mathrm{Im}^r A^s = 0$$

proof Since colim_s : exact,

$$\mathrm{colim}_s K_\infty\mathrm{Im}^r A^s = \mathrm{colim}_s\big(\mathrm{Ker}(\mathrm{Im}^r A^s \longrightarrow \overset{\mathrm{colim}_s A^s}{\overset{\|}{A^{-\infty}}})\big)$$
$$= \mathrm{Ker}(\mathrm{colim}_s \mathrm{Im}^r A^s \longrightarrow \mathrm{colim}_s A^s)$$
$$= \mathrm{colim}_s\big(\underbrace{\mathrm{Ker}(\mathrm{Im}^r A^s \longrightarrow A^s)}_{0}\big)$$
$$= 0$$

//

proof of Lem 3.6.5

Consider

$$0 \to K_n \mathrm{Im}^r A^s \to K_\infty \mathrm{Im}^r A^s \xrightarrow{i^n} K_\infty \mathrm{Im}^{r+n} A^{s-n} \to 0 \qquad \circledast$$

exact

(1) Apply $\lim_r$ to $\circledast$

$$0 \to \lim_r K_n \mathrm{Im}^r A^s \to \lim_r K_\infty \mathrm{Im}^r A^s \to \overset{K_\infty \mathrm{Im}^{\omega} A^{s-n}\ \overset{\leftarrow \text{Lem3.6.3(1)}}{\shortparallel}}{\lim_r K_\infty \mathrm{Im}^{r+n} A^{s-n}}$$
$$\to R\lim_r K_n \mathrm{Im}^r A^s \to R\lim_r K_\infty \mathrm{Im}^r A^s \to R\lim_r K_\infty \mathrm{Im}^{r+n} A^{s-n} \to 0$$

exact

$\Big\downarrow \underset{n}{\mathrm{colim}}$

$$0 \to \text{(i)} \to \text{(ii)} \to \overset{0\ \overset{\leftarrow \text{Lem 3.6.6}}{\shortparallel}}{\underset{n}{\mathrm{colim}}\, K_\infty \mathrm{Im}^{\omega} A^{s-n}}$$
$$\to \underset{n}{\mathrm{colim}}\, R\lim_r K_n \mathrm{Im}^r A^s \to \cancel{\underset{n}{\mathrm{colim}}}\, R\lim_r K_\infty \mathrm{Im}^r A^s \xleftarrow{} \text{indep of } n$$
$$\to \underbrace{\underset{n}{\mathrm{colim}}\, R\lim_r K_\infty \mathrm{Im}^{r+n} A^{s-n}}_{\overset{\shortparallel}{W}} \to 0$$

(2) By $\circledast$, we have

$$K_\infty \mathrm{Im}^{r+n} A^{s-n} \cong \frac{K_\infty \mathrm{Im}^r A^s}{K_n \mathrm{Im}^r A^s}$$

$\Big\downarrow R\lim_r$

$$R\lim_r K_\infty \mathrm{Im}^{r+n} A^{s-n} \cong R\lim_r \frac{K_\infty \mathrm{Im}^r A^s}{K_n \mathrm{Im}^r A^s}$$
$$\mathrel{\rotatebox{90}{$\cong$}}$$
$$R\lim_r K_\infty \mathrm{Im}^r A^{s-n}$$

$\Big\downarrow \underset{n}{\mathrm{colim}}$

$$\underset{n}{\mathrm{colim}}\, R\lim_r K_\infty \mathrm{Im}^r A^{s-n} \cong \underset{n}{\mathrm{colim}}\, R\lim_r \frac{K_\infty \mathrm{Im}^r A^s}{K_n \mathrm{Im}^r A^s}$$
$$\mathrel{\rotatebox{90}{$\cong$}}$$
$$\underset{s}{\mathrm{colim}}\, R\lim_r K_\infty \mathrm{Im}^r A^s = W$$

//

Define

$$F^s A^{-\infty} := \mathrm{Im}(\eta^s : A^s \to A^{-\infty})$$

$\rightsquigarrow \{F^s A^{-\infty}\}_s$: filtration on $A^{-\infty}$

(see Def 4.1.15)

これの Lem により、

もし $W=0$ なら $\lim_s F^s A^{-\infty}$, $R\lim_s F^s A^{-\infty}$ が 分かり易くなる

Lem 3.6.7 [Boa, Lem 8.11]

$$0 \to \underset{s}{\mathrm{colim}}\, Q^s \to \lim_s F^s A^{\infty} \to W$$
$$\to \underset{s}{\mathrm{colim}}\, RQ^s \to R\lim_s F^s A^{-\infty} \to 0 \quad : \text{exact}$$

proof

$$0 \to K_\infty \mathrm{Im}^r A^s \to \mathrm{Im}^r A^s \to F^{r+s} A^{-\infty} \to 0 \quad : \text{exact}$$

$\Big\downarrow \lim_r$

$$0 \to \overset{K_\infty \mathrm{Im}^{\omega} A^s\ \overset{\leftarrow \text{Lem3.6.3(1)}}{\shortparallel}}{\lim_r K_\infty \mathrm{Im}^r A^s} \to \overset{Q^s\ \overset{}{\shortparallel}}{\lim_r \mathrm{Im}^r A^s} \to \overset{\lim_s F^s A^{-\infty}\ \overset{}{\shortparallel}}{\lim_r F^{r+s} A^{\infty}}$$
$$\to R\lim_r K_\infty \mathrm{Im}^r A^s \to \underset{\underset{RQ^s}{\shortparallel}}{R\lim_r \mathrm{Im}^r A^s} \to \underset{\underset{R\lim_s F^s A^{-\infty}}{\shortparallel}}{R\lim_r F^{r+s} A^{-\infty}} \to 0$$

exact

$\Big\downarrow \underset{s}{\mathrm{colim}}$

The exact seq in the statement

(∵ By Lem 3.6.6, $\underset{s}{\mathrm{colim}}\, K_\infty \mathrm{Im}^{\omega} A^s = 0$)

//

$A^{-\infty} = 0$ なら、W は簡単になる：

Lem 3.6.8 [Boa, Lem 8.14]

Assume $A^{-\infty} = 0$

Then $W = \underset{s}{\mathrm{colim}}\, RQ^s$

proof

$$K_\infty \mathrm{Im}^r A^s = \mathrm{Ker}(\mathrm{Im}^r A^s \to \underset{\underset{0}{\shortparallel}}{A^{-\infty}}) = \mathrm{Im}^r A^s$$

$$\rightsquigarrow W = \underset{s}{\mathrm{colim}}\, \underset{\underset{RQ^s}{\shortparallel}}{R\lim_r \mathrm{Im}^r A^s}$$

//

$\{K_n \mathrm{Im}^r A^s\}_{r,n}$: "double filtration" of A^s

とその minimal subquotient と W の関係として

次が成立:

Lem 3.6.9 [Boa, in the proof of Lem 8.1]

Fix $1 \leq r_0 < \omega$, $s \in \mathbb{Z}$

Assume

$1 \leq {}^\forall n < \infty$, $r_0 \leq {}^\forall r < \omega$,

$$\frac{K_n \mathrm{Im}^r A^s}{K_n \mathrm{Im}^{r+1} A^s + K_{n-1} \mathrm{Im}^r A^s} = 0$$

Then

(1) $0 \leq {}^\forall n \leq \infty$, $r_0 \leq {}^\forall r < \omega$,

$$K_n \mathrm{Im}^r A^s = K_n \mathrm{Im}^{r+1} A^s$$

(2) $W = 0$

proof

(1) induction on $n < \infty$

$n=0$ $K_0 \mathrm{Im}^r A^s = K_0 \mathrm{Im}^{r+1} A^s = 0$

$n \geq 1$

$$K_n \mathrm{Im}^r A^s \overset{\text{assump}}{=} K_n \mathrm{Im}^{r+1} A^s + \overset{\substack{K_{n-1} \mathrm{Im}^{r+1} A^s \\ \parallel \leftarrow \text{ind. hyp.}}}{\boxed{K_{n-1} \mathrm{Im}^r A^s}}$$

$$= K_n \mathrm{Im}^{r+1} A^s$$

For $n = \infty$,

$$K_\infty \mathrm{Im}^r A^s = \bigcup_m K_m \mathrm{Im}^r A^s \overset{\text{上で示した}}{=} \bigcup_m K_m \mathrm{Im}^{r+1} A^s = K_\infty \mathrm{Im}^{r+1} A^s$$

(2) By Lem 3.6.5 (2),

$$W \cong \underset{n}{\operatorname{colim}} \underbrace{\underset{r}{R\lim} \overset{\text{indep. of } r (\geq r_0)}{\boxed{\frac{K_\infty \mathrm{Im}^r A^s}{K_n \mathrm{Im}^r A^s}}}}_{\substack{\parallel \\ 0}} = 0$$

//

Example 3.6.10 [Boa, Example in p.26]

For $n \geq 0$, define

$$A^n = A^{-n} := \mathbb{K}\{x_t \mid t \geq n\} = \bigoplus_{t \geq n} \mathbb{K} x_t \quad \leftarrow \text{free mod}/\mathbb{K}$$

$$i : A^{n+1} \longrightarrow A^n$$
$$x_t \longmapsto x_t$$

$$i : A^{-n} \longrightarrow A^{-n-1}$$
$$x_t \longmapsto x_t \quad (t \geq n+1)$$
$$x_n \longmapsto x_{n+1}$$

$$\begin{tikzcd}
\cdots \arrow[r] & A^3 \arrow[r] & A^2 \arrow[r] & A^1 \arrow[r] & A^0 \arrow[r] & A^{-1} \arrow[r] & A^{-2} \arrow[r] & A^{-3} \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
& x_3 \arrow[r, mapsto] & x_3 \arrow[r, mapsto] & x_3 \arrow[r, mapsto] & x_3 \arrow[r, mapsto] & x_3 \arrow[r, mapsto] & x_3 \arrow[r, mapsto] & x_3 & \\
& & x_2 \arrow[r, mapsto] & x_2 \arrow[r, mapsto] & x_2 \arrow[r, mapsto] & x_2 \arrow[r, mapsto] & x_2 \arrow[ur, mapsto] & & \\
& & & x_1 \arrow[r, mapsto] & x_1 \arrow[r, mapsto] & x_1 \arrow[ur, mapsto] & & & \\
& & & & x_0 \arrow[ur, mapsto] & & & &
\end{tikzcd}$$

Then we have:

- $A^{-\infty} = \mathbb{K} x \neq 0$

  $$\eta^s : A^s \longrightarrow A^{-\infty} \qquad ({}^\forall s, t)$$
  $$x_t \longmapsto x$$

- ${}^\forall s$, $F^s A^{\infty} = A^{-\infty}$

  ($\because$ ${}^\forall s$, η^s : surj)

- $F^\infty A^{-\infty} = A^{-\infty} \neq 0$, $RF^\infty A^\infty = 0$

- $A^\infty = 0$, $RA^\infty = \prod \mathbb{K} x_t / \bigoplus \mathbb{K} x_t$

  ($\because$ Ex 3.2.9)

- ${}^\forall s$, $Q^s = 0$

  ($\because$ direct computation)

- ${}^\forall n \geq 0$, $RQ^n \cong RQ^{-n} \cong \prod_{t \geq n} \mathbb{K} x_t \Big/ \bigoplus_{t \geq n} \mathbb{K} x_t$

  ($\because$ Ex 3.2.9)

- $W \neq 0$

  ($\because$ By Lem 3.6.7,

  $$0 \to \underset{s}{\operatorname{colim}} \underbrace{Q^s}_{\substack{\parallel \\ 0}} \to F^\infty A^{-\infty} \xrightarrow[\text{inj}]{} W \quad : \text{exact}$$

)

§3.7 Ordinals and image subsequences

(このsubsectionの内容は他のところでは使わないので、とばしても大丈夫)

Ref

[Cie] Ciesielski, Set theory for the working mathematician, Chapter 4

↖ ordinalの基本的なことが書いてある

Aim

$\mathrm{Im}^r A^s$ の r を ordinal に一般化したい

Def 3.7.1

- α: set に対して、

α: ordinal (number)

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet\ \forall \beta \in \alpha,\ \beta \subset \alpha \\ \bullet\ \forall \beta, \gamma \in \alpha, \\ \qquad \beta = \gamma \text{ or } \beta \in \gamma \text{ or } \gamma \in \beta \end{cases}$$

- α, β: ordinal に対して、

$$\alpha < \beta \underset{\text{def}}{\Longleftrightarrow} \alpha \in \beta$$

Ex 3.7.2

$0 = \emptyset$

$1 = \{\emptyset\} = \{0\}$

$2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$

$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$

$\vdots$

$n = \{0, 1, 2, \cdots, n-1\}$

$\vdots$

$\omega = \mathbb{N} = \{0, 1, 2, \cdots\}$

$\omega + 1 = \{0, 1, 2, \cdots, \omega\}$

以下、いつものように

$A = \{A^s\}_{s \in \mathbb{Z}}$: seq of $\mathbb{K}$-mods

(i.e. $\cdots \xrightarrow{i} A^s \xrightarrow{i} A^{s-1} \longrightarrow \cdots$)

とする。

Def 3.7.3

- Define

$\mathrm{Im}\, A$: first image subsequence

by

$(\mathrm{Im}\, A)^s := \mathrm{Im}^1 A^s$ as in previous sections

- For α: ordinal, define $\mathrm{Im}^\alpha A$: seq by

$$\mathrm{Im}^\alpha A := \begin{cases} A & (\alpha = 0) \\ \mathrm{Im}(\mathrm{Im}^\beta A) & (\alpha = \beta + 1 : \text{successor ordinal}) \\ \bigcap_{\beta < \alpha} \mathrm{Im}^\beta A & (\alpha : \text{limit ordinal}) \end{cases}$$

$$\left(\begin{tikzcd} i : (\mathrm{Im}^\alpha A)^s \arrow[r] \arrow[d, phantom, sloped, "\subset"] & (\mathrm{Im}^\alpha A)^{s-1} \arrow[d, phantom, sloped, "\subset"] \\ A^s \arrow[r, "i"] & A^{s-1} \end{tikzcd}\right)$$

Write

$\mathrm{Im}^\alpha A^s := (\mathrm{Im}^\alpha A)^s$

Rmk 3.7.4

- 上の def で well-defd なのは、ordinal の基本性質 (see [Cie, Thm 4.3.1])
- $\alpha = r \in \mathbb{N}$ のときは、前に def したものと同じになっている
- $A^s \supset \mathrm{Im}^1 A^s \supset \mathrm{Im}^2 A^s \supset \cdots \supset \mathrm{Im}^\omega A^s \supset \mathrm{Im}^{\omega+1} A^s \supset \cdots$

Lem 3.7.5

$$\boxed{\mathrm{Im}^\omega A^s = Q^s}$$

~~proof~~ def が全く同じ //

⑭ image order

Lem 3.7.6

(1) $\exists \alpha$: ordinal s.t. $\mathrm{Im}^{\alpha} A = \mathrm{Im}^{\alpha+1} A$

(2) α: as above
Then $\forall \beta \geq \alpha$, $\mathrm{Im}^{\beta} A = \mathrm{Im}^{\alpha} A$

proof

(1) $\forall \alpha$, $\mathrm{Im}^{\alpha} A \neq \mathrm{Im}^{\alpha+1} A$ と仮定する. ── ①

(i.e. $\bigoplus_s \mathrm{Im}^{\alpha} A^s \supsetneq \bigoplus_s \mathrm{Im}^{\alpha+1} A^s$)

Take α_0: ordinal s.t.

$$\#\alpha_0 > \#(\mathcal{P}(\bigoplus_s A^s)) \quad \text{── ②}$$

power set

(∵ 整列可能定理より,
$\exists$ well-order on $\mathcal{P}(\mathcal{P}(\bigoplus_s A^s))$
$\rightsquigarrow$ $\alpha_0 :=$ (its order type))

Then we have a map

$$f : \alpha_0 = \{\alpha < \alpha_0\} \longrightarrow \mathcal{P}(\bigoplus_s A^s)$$
$$\alpha \longmapsto \bigoplus_s \mathrm{Im}^{\alpha} A^s$$

By ①, we have

$$\forall \alpha > \forall \beta, \quad f(\alpha) \subsetneq f(\beta)$$

$\rightsquigarrow$ $f : \alpha_0 \longrightarrow \mathcal{P}(\bigoplus_s A^s)$: inj

$\rightsquigarrow$ ② に矛盾

(2) Fix such α.
By transfinite ind on $\beta \geq \alpha$, we prove

$$\mathrm{Im}^{\beta} A = \mathrm{Im}^{\alpha} A$$

$\underline{\beta = \alpha}$ obvious

$\underline{\beta > \alpha}$

Case 1 $\beta = \gamma + 1$: successor

$$\mathrm{Im}^{\beta} A = \mathrm{Im}(\mathrm{Im}^{\gamma} A) = \mathrm{Im}(\mathrm{Im}^{\alpha} A)$$
$$= \mathrm{Im}^{\alpha+1} A = \mathrm{Im}^{\alpha} A$$

ind. hyp.

Case 2 β : limit ordinal

$$\mathrm{Im}^{\beta} A = \bigcap_{\gamma < \beta} \underbrace{\mathrm{Im}^{\gamma} A}_{\mathrm{Im}^{\alpha} A \text{ by ind hyp.}} = \mathrm{Im}^{\alpha} A$$

//

Def 3.7.7

$\sigma := \min\{\alpha : \text{ordinal} \mid \mathrm{Im}^{\alpha} A = \mathrm{Im}^{\alpha+1} A\}$
image order of A

(∵ Take α as in Lem 3.7.6 (1)
Since α: well-ordered,
(i.e. $\forall S \subset \alpha$, $\exists \min S \in S$)
$\sigma := \min\{\gamma \in \alpha \mid \mathrm{Im}^{\gamma} A = \mathrm{Im}^{\gamma+1} A\}$
is well-defd)

Prop 3.7.8

σ : image order of A
Then
$\forall \alpha \geq \sigma$, $\mathrm{Im}^{\alpha} A = \mathrm{Im}^{\sigma} A$

proof Lem 3.7.6 (2) //

一般の(ωよりも大きな) ordinal に拡張して意味があるのか疑問に思うかもしれないが, 次の Prop より, ちゃんと意味がある

Prop 3.7.9 [Boa, Example in p. 10]

$\forall \sigma$: ordinal,
$\exists A$: sequence
s.t. (image order of A) $= \sigma$

proof Define X^s: set by

- For $s < 0$, $X^s := \emptyset$
- For $s \geq 0$,

$$X^s := \{(\alpha_s, \alpha_{s-1}, \cdots, \alpha_0) \mid \alpha_i : \text{ordinal s.t. } 0 \leq \alpha_s < \alpha_{s-1} < \cdots < \alpha_0 < \sigma\}$$
$$(\subset \sigma^{\times s})$$

Define A by

- $A^s := \mathbb{K} X^s$ (free mod on X^s)
- $i : A^s \longrightarrow A^{s-1}$
  $(\alpha_s, \cdots, \alpha_0) \longmapsto (\alpha_{s-1}, \cdots, \alpha_0)$

Then, by transfinite ind. on α, we have

$$\mathrm{Im}^{\alpha} A^s = \mathbb{K}\{(\alpha_s, \cdots, \alpha_0) \in X^s \mid \alpha_s \geq \alpha\}$$

$\rightsquigarrow$ $\mathrm{Im}^{\sigma} A^s = 0$, $\mathrm{Im}^{\alpha} A^s \neq 0$ for $\forall \alpha < \sigma$

$\rightsquigarrow$ (image order of A) $= \sigma$

//

以前は $\mathrm{Im}^\sigma A^s := \mathrm{Im}\,\varepsilon^s$ と定義していた

このdefとcompatibleであることを示す

Prop3.7.10

> α : image order of A
>
> Then
>
> $$\mathrm{Im}^\sigma A^s = \mathrm{Im}(\varepsilon^s : A^\infty \to A^s)$$

proof

$\mathrm{Im}^\sigma A^s \supset \mathrm{Im}\,\varepsilon^s$

By transfinite induction on α, we prove

$$\mathrm{Im}^\alpha A^s \supset \mathrm{Im}\,\varepsilon^s$$

$\alpha = 0$ 明らか!

$\alpha > 0$

Case1 $\alpha = \beta + 1$: successor のとき

$$\begin{aligned}\mathrm{Im}^\alpha A^s &= (\mathrm{Im}(\mathrm{Im}^\beta A))^s \\ &= \mathrm{Im}(i : \mathrm{Im}^\beta A^{s+1} \to \mathrm{Im}^\beta A^s) \\ &= i(\mathrm{Im}^\beta A^{s+1}) \\ &\underset{\substack{\uparrow \\ \text{ind. hyp.}}}{\supset} i(\mathrm{Im}\,\varepsilon^{s+1}) = \mathrm{Im}(i \circ \varepsilon^{s+1}) \\ &\qquad\qquad\qquad = \mathrm{Im}\,\varepsilon^s\end{aligned}$$

Case2 α : limit ordinal のとき

$$\mathrm{Im}^\alpha A^s = \bigcap_{\beta<\alpha} \mathrm{Im}^\beta A^s \underset{\substack{\uparrow \\ \text{ind hyp}}}{\supset} \bigcap_{\beta<\alpha} \mathrm{Im}\,\varepsilon^s = \mathrm{Im}\,\varepsilon^s$$

$\mathrm{Im}^\sigma A^s \subset \mathrm{Im}\,\varepsilon^s$

$\forall t,\ i : \mathrm{Im}^\sigma A^{t+1} \twoheadrightarrow \mathrm{Im}^\sigma A^t$: surj

$$\left(\because\ i(\mathrm{Im}^\sigma A^{t+1}) \underset{\substack{\uparrow \\ \text{def of } \mathrm{Im}^\sigma}}{=} \mathrm{Im}^{\sigma+1} A^t \underset{\substack{\uparrow \\ \text{def of } \sigma}}{=} \mathrm{Im}^\sigma A^t \right)$$

Hence, for $\forall x \in \mathrm{Im}^\sigma A^s$,

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Im}^\sigma A^{s+2} \arrow[r] & \mathrm{Im}^\sigma A^{s+1} \arrow[r] & \mathrm{Im}^\sigma A^s \\ \cdots \arrow[r, mapsto] & \exists \arrow[r, mapsto] & \exists \arrow[r, mapsto] & x \end{tikzcd}$$

This seq gives an elem in $\varprojlim_t A^t$

$\rightsquigarrow x \in \mathrm{Im}\,\varepsilon^s$

//

◎ Generalization of properties of Q^s, $\mathrm{Im}^\sigma A^s$

Thm3.5.3(a), Lem4.1.16 の一般化:

Prop3.7.11

> Fix α : ordinal
>
> $\mathrm{Im}^\alpha A^s \hookrightarrow A^s$ induces
>
> $$\varprojlim_s \mathrm{Im}^\alpha A^s \xrightarrow{\cong} \varprojlim_s A^s$$

proof

By Prop 3.7.10,

$$\varprojlim_t A^t \xrightarrow{\varepsilon^s} \mathrm{Im}^\sigma A^s \subset \mathrm{Im}^\alpha A^s$$

$$\overset{\varprojlim_s}{\rightsquigarrow} \quad \varprojlim_t A^t \longrightarrow \varprojlim_s \mathrm{Im}^\alpha A^s$$

This gives the inverse.

//

Prop 3.5.18の一般化

Prop 3.7.12

> $f : \{A^s\} \to \{\bar{A}^s\}$: morph of seqs
>
> Fix $\begin{cases} \cdot\ s_0 \in \mathbb{Z} \\ \cdot\ \alpha_0 : \text{ordinal} \end{cases}$
>
> Assume
>
> $\forall s \geq s_0,\ f : \mathrm{Im}^{\alpha_0} A^s \xrightarrow{\cong} \mathrm{Im}^{\alpha_0} \bar{A}^s$: isom
>
> Then
>
> $\forall \alpha \geq \alpha_0,\ \forall s \geq s_0,$
>
> $$f : \mathrm{Im}^\alpha A^s \xrightarrow{\cong} \mathrm{Im}^\alpha \bar{A}^S \text{ : isom}$$

proof transfinite ind on α

$\alpha = \alpha_0$ 自明

$\alpha > \alpha_0$

Case1 $\alpha = \beta + 1$: successor のとき

$$\begin{tikzcd} \mathrm{Im}^\beta A^{s+1} \arrow[r, "f", "\cong"'] \arrow[d, "i"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Im}^\beta \bar{A}^{s+1} \arrow[d, "i"] \\ \mathrm{Im}^\beta A^s \arrow[r, "f", "\cong"'] & \mathrm{Im}^\beta \bar{A}^s \end{tikzcd}$$

(assump が $\forall s \geq s_0$ だから)

ind. hyp.

$$\rightsquigarrow \mathrm{Im}^\alpha A^s = i(\mathrm{Im}^\beta A^{s+1}) = i(\mathrm{Im}^\beta \bar{A}^{s+1}) = \mathrm{Im}^\alpha \bar{A}^s$$

Case2 α : limit ordinal のとき

$$\mathrm{Im}^\alpha A^s = \bigcap_{\beta<\alpha} \mathrm{Im}^\beta A^s \xrightarrow[\cong]{f} \bigcap_{\beta<\alpha} \mathrm{Im}^\beta \bar{A}^s = \mathrm{Im}^\alpha \bar{A}^s$$

ind hyp.

//

§4. Convergence [Boa, Part II]

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & A^{s+1} \arrow[rr, "i"] \arrow[dl] & & A^{s} \arrow[rr, "i"] \arrow[dl, "j"'] & & A^{s-1} \arrow[r] \arrow[dl, "j"'] & \cdots \\
{} & & E^{s} \arrow[ul, "k"] & & E^{s-1} \arrow[ul, "k"] & & {} \arrow[ul]
\end{tikzcd}$$

unrolled exact couple

Rmk

以下の議論は全て degree-wise で良い

E_r^{st}

§4.1 Types of convergence [Boa, §5]

Recall

- We have a filtration of $E^s = E_1^s$:

$0 = B_1^s \subset B_2^s \subset \cdots \subset B_\infty^s \subset \operatorname{Im} j = \operatorname{Ker} k \subset Z_\infty^s \subset \cdots \subset Z_2^s \subset Z_1^s = E^s$

where

- $Z_r^s := k^{-1}(\operatorname{Im}^{r-1} A^{s+1})$
  $B_r^s := j(K_{r-1} A^s)$ $\quad (1 \leq r < \infty)$
- $Z_\infty^s := \bigcap_r Z_r^s, \quad B_\infty^s := \bigcup_r B_r^s$

- $E_r^s := Z_r^s / B_r^s \quad (1 \leq r \leq \infty)$

introduced by the "policy"

Def 4.1.1

$$RE_\infty^s := R\lim_r Z_r^s$$

$RE_\infty = 0$ だと嬉しい

Lem 4.1.2

Fix r_0 with $1 \leq r_0 < \infty$

Then

(1) $E_\infty^s \cong \dfrac{Z_\infty^s / B_{r_0}^s}{B_\infty^s / B_{r_0}^s}$

(2) $RE_\infty^s \cong R\lim_r \left(Z_r^s / B_{r_0}^s \right)$

where $\cdots \hookrightarrow Z_r^s / B_{r_0}^s \hookrightarrow Z_{r-1}^s / B_{r_0}^s \hookrightarrow \cdots \hookrightarrow Z_{r_0}^s / B_{r_0}^s$

proof (1) 明らか. (2) Prop 3.4.3 (b) //

Rmk 4.1.3

Lem 4.1.2 shows that

E_∞ and RE_∞ depend only on E_r for $r \geq r_0$

と[Boa]に書いてある.

$\rightsquigarrow$ Prop 4.1.4, Prop 4.1.9

Prop 4.1.4

Fix s.

Assume

$\exists r_0, \ \forall r \geq r_0, \ d_r^s = 0 : E_r^s \to E_r^{s+r}$

Then

$RE_\infty^s = 0$

proof

By Cor 2.2.10 (1), we have

$Z_\infty^s = \cdots = Z_{r_0+1}^s = Z_{r_0}^s$

$\rightsquigarrow RE_\infty^s = R\lim_r Z_r^s \underset{\text{Prop 3.2.5 (b)}}{=} 0$ //

$\rightsquigarrow$ The condition $RE_\infty = 0$ is "internal to s.s."

簡単な十分条件を与えておく.

Def 4.1.5

M : $\mathbb{K}$-mod

M satisfies descending chain condition

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall$ filtration $M \supset M^1 \supset M^2 \supset \cdots$

$\exists n_0, \ \forall n \geq n_0, \ M^n = M^{n_0}$

Prop 4.1.6

Fix s

Assume

$\exists r_0$, $E_{r_0}^s$ satisfies descending chain cond.

Then

$\exists r_1, \ \forall r \geq r_1, \ d_r^s = 0$

proof

$\cdots \subset Z_r^s / B_{r_0}^s \subset \cdots \subset Z_{r_0+1}^s / B_{r_0}^s \subset Z_{r_0}^s / B_{r_0}^s = E_{r_0}^s$

By assump,

$\exists r_1, \ \underbrace{\forall r \geq r_1, \ Z_r^s / B_{r_0}^s = Z_{r_1}^s / B_{r_0}^s}$

$\Longleftrightarrow Z_{r_1}^s = Z_{r_1+1}^s = \cdots$

$\underset{\text{Cor 2.2.9 (1)}}{\rightsquigarrow} \forall r \geq r_1, \ d_r^s = 0$ //

Lem 4.1.7

M satisfies desc. chain cond. in the following cases

(1) $\mathbb{K}$: field, M: fin dim$/\mathbb{K}$

(2) $\mathbb{K}=\mathbb{Z}$, M: finite ab. grp.

← $\mathbb{K}=\mathbb{Z}$, $M=\mathbb{Z}$ はダメ

Cor 4.1.8

Fix s

Assume

$\exists r_0$. $E^s_{r_0}$ satisfies (1) or (2) in Lem 4.1.7

Then

$RE^s_\infty = 0$

次に、spectral seq の comparison について:

Prop 4.1.9

$f:(A,E)\longrightarrow(\bar{A},\bar{E})$: morphism of unrolled exact couple

Assume

$\exists r_0$, $f_{r_0}: E_{r_0}\xrightarrow{\cong}\bar{E}_{r_0}$: isom

Then

(1) $r_0\le{}^\forall r<\infty$, $f_r: E_r\xrightarrow{\cong}\bar{E}_r$: isom

(2) $f_\infty: E_\infty\xrightarrow{\cong}\bar{E}_\infty$: isom

(3) $Rf_\infty: RE_\infty\xrightarrow{\cong}R\bar{E}_\infty$: isom

proof (1) $E_{r+1}=H(E_r,d_r)$ より ok.

(2)

$r_0\le{}^\forall r<\infty$, $B_{r+1}/B_r\xrightarrow{\cong}\bar{B}_{r+1}/\bar{B}_r$ —— ①

$$\begin{tikzcd} B_{r+1}/B_r \arrow[d, equal] & \bar{B}_{r+1}/\bar{B}_r \arrow[d, equal] \\ \text{(∵ } \operatorname{Im} d_r \arrow[r, "\boxed{\cong}"] & \operatorname{Im}\bar{d}_r \text{ by Prop2.2.7(2))} \end{tikzcd}$$

(1)より

$r_0\le{}^\forall r<\infty$, $Z_r/B_{r_0}\xrightarrow{\cong}\bar{Z}_r/\bar{B}_{r_0}$ —— ②

(∵ By downward ind. on m, we prove

$r_0\le{}^\forall m\le r$, $Z_r/B_m\xrightarrow{\cong}\bar{Z}_r/\bar{B}_m$

$\underline{m=r}$ $E_r\xrightarrow{\cong}\bar{E}_r$ なので ok

$\underline{m<r}$

$$\begin{tikzcd} 0 \arrow[r] & B_{m+1}/B_m \arrow[r] \arrow[d, "① \cong"'] & Z_r/B_m \arrow[r] \arrow[d] & Z_r/B_{m+1} \arrow[r] \arrow[d, "\cong \text{ by ind. hyp.}"] & 0 : \text{exact} \\ 0 \arrow[r] & \bar{B}_{m+1}/\bar{B}_m \arrow[r] & \bar{Z}_r/\bar{B}_m \arrow[r] & \bar{Z}_r/\bar{B}_{m+1} \arrow[r] & 0 : \text{exact} \end{tikzcd}$$
)

Hence

$Z_\infty/B_{r_0}\xrightarrow{\cong}\bar{Z}_\infty/\bar{B}_{r_0}$ —— ③

$$\begin{tikzcd} Z_\infty/B_{r_0} \arrow[d, equal] & & \bar{Z}_\infty/\bar{B}_{r_0} \arrow[d, equal] \\ \text{(∵ } \lim_r Z_r/B_{r_0} \arrow[rr, "\boxed{\cong}", "②"'] & & \lim_r \bar{Z}_r/\bar{B}_{r_0} \text{)} \end{tikzcd}$$

← Prop3.4.3(b) →

Similarly to ②, we have

$r_0\le{}^\forall r<\infty$, $B_r/B_{r_0}\xrightarrow{\cong}\bar{B}_r/\bar{B}_{r_0}$

(∵ By downward ind. on $r_0\le m\le r$, we prove

$B_r/B_m\xrightarrow{\cong}\bar{B}_r/\bar{B}_m$

$\underline{m=r}$ $0\xrightarrow{\cong}0$ 自明.

$\underline{m<r}$

$$\begin{tikzcd} 0 \arrow[r] & B_{m+1}/B_m \arrow[r] \arrow[d, "① \cong"'] & B_r/B_m \arrow[r] \arrow[d] & B_r/B_{m+1} \arrow[r] \arrow[d, "\cong \text{ ind. hyp.}"] & 0 \\ 0 \arrow[r] & \bar{B}_{m+1}/\bar{B}_m \arrow[r] & \bar{B}_r/\bar{B}_m \arrow[r] & \bar{B}_r/\bar{B}_{m+1} \arrow[r] & 0 \end{tikzcd}$$
)

Hence

$B_\infty/B_{r_0}\xrightarrow{\cong}\bar{B}_\infty/\bar{B}_{r_0}$ —— ④

$$\begin{tikzcd} B_\infty/B_{r_0} \arrow[d, equal] \\ \text{(∵ } \operatorname{colim}_r B_r/B_{r_0} \text{ by Prop3.4.3(a))} \end{tikzcd}$$

By ③ and ④, we have

$E_\infty\xrightarrow{\cong}\bar{E}_\infty$

$$\begin{tikzcd} E_\infty \arrow[d, equal] \\ \text{(∵ } \dfrac{Z_\infty/B_{r_0}}{B_\infty/B_{r_0}} \text{ by Lem 4.1.2(1))} \end{tikzcd}$$

(3) ② と Lem 4.1.2(2) より ok

//

spectral seq の target と convergence を定義する.

Def 4.1.10

G : (graded) $\mathbb{K}$-mod is a <u>target</u> of $\{E_r\}$

$\overset{\text{def}}{\Longleftrightarrow}$ it is equipped with
- $\{F^s\} = \{F^s G\}$: filtration of G
- $\left\{ F^s / F^{s+1} \longrightarrow E^s_\infty \right\}_s$: family of $\mathbb{K}$-lin. maps

← [Boa] では少しあいまいな書き方をしていたので独自解釈した

← We write

$$E^s_1 \Rightarrow G$$

$(\text{or } E^s_2 \Rightarrow G, \text{ etc} \cdots)$

We need to state "convergence" separately

Def 4.1.11 [Boa, Def 5.2]

G : target of $\{E_r\}$

(1) $\{E_r\}$ <u>converges weakly</u> to G

$\overset{\text{def}}{\Longleftrightarrow}$
- $\{F^s\}$ exhausts G ($\underline{\text{i.e.}}$ $F^{-\infty} = G$)
- $\forall s.\ F^s/F^{s+1} \xrightarrow{\cong} E^s_\infty$

(2) $\{E_r\}$ <u>converges</u> to G

$\overset{\text{def}}{\Longleftrightarrow}$
- (1)
- $\{F^s\}$ is Hausdorff ($\underline{\text{i.e.}}$ $F^\infty = 0$)

(3) $\{E_r\}$ <u>converges strongly</u> to G

$\overset{\text{def}}{\Longleftrightarrow}$
- (2)
- $\{F^s\}$ is complete ($\underline{\text{i.e.}}$ $RF^\infty = 0$)

Rmk 4.1.12

- Def 4.1.11 is the terminology of Cartan-Eilenberg
- strong convergence $\Rightarrow$ recover G up to extension
  ↑ Prp 3.4.5

  ↪ strong convergence まで言えれば文句なし

Comparison theorem :

Thm 4.1.13 [Boa, Thm 5.3]

- $f : (A, E) \longrightarrow (\bar{A}, \bar{E})$: morph of unrolled exact couples
- G : target of $\{E_r\}$

  $\bar{G}$: target of $\{\bar{E}_r\}$
- $g : G \longrightarrow \bar{G}$: morph of filtered modules

<u>Assume</u>

(1) $\{E_r\}$ converges strongly to G

$\{\bar{E}_r\}$ converges to $\bar{G}$ ← not necessarily strongly

(2) f_∞ and g are compatible :

$$\begin{tikzcd} F^s/F^{s+1} \arrow[r, "g"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \bar{F}^s/\bar{F}^{s+1} \arrow[d, "\cong"] \\ E^s_\infty \arrow[r, "f_\infty"] & \bar{E}^s_\infty \end{tikzcd}$$

(3) $\exists r_0 \leq \infty,\ f_{r_0} : E_{r_0} \xrightarrow{\cong} \bar{E}_{r_0}$: isom

<u>Then</u>

$g : G \xrightarrow{\cong} \bar{G}$: isom of filtered modules

$\left(\underline{\text{i.e.}} \quad g : G \xrightarrow{\cong} \bar{G} \quad \forall s.\ F^s \xrightarrow{\cong} \bar{F}^s \right)$

proof

(3) $\overset{\text{Prp 4.1.9}}{\leadsto}$ f_∞ : isom

$\overset{(2)}{\leadsto}$ $\forall s.\ g : F^s/F^{s+1} \xrightarrow{\cong} \bar{F}^s/\bar{F}^{s+1}$: isom

$\underset{\text{Thm 3.4.6}}{\leadsto}$ g : isom of filtered modules

← assump (1)

//

Rmk 4.1.14

この note では §4 の最初から unrolled exact couple を仮定しているが, ここまで (Thm 4.1.13 まで) の議論は「一般の spectral sequence」でもできる.

(r_0 を fix して, $0 < B^s_r \subset Z^s_r \subset E^s_{r_0}$ を def する.)

[Boa] ではそうしているけど.

そんな一般化して意味ある?

Two filtered groups

Two candidates of target:

$A^{-\infty}$ and A^{∞}

Def 4.1.15

- $F^sA^{-\infty} := \operatorname{Im}(\eta^s : A^s \to A^{-\infty})$
- $F^sA^{\infty} := \operatorname{Ker}(\varepsilon^s : A^\infty \to A^s)$

$\operatorname{Im}^\sigma A^s := \operatorname{Im}(\varepsilon^s : A^\infty \to A^s)$

(c.f. image order, §3.6 (?))

Lem 4.1.16

$A^\infty \cong \lim_s \operatorname{Im}^\sigma A^s$

which is given by

- $\varepsilon^s : A^\infty \to \operatorname{Im}^\sigma A^s$

  $\rightsquigarrow A^\infty \to \lim_s \operatorname{Im}^\sigma A^s$

- $\operatorname{Im}^\sigma A^s \hookrightarrow A^s$

  $\rightsquigarrow \lim_s \operatorname{Im}^\sigma A^s \to \lim_s A^s = A^\infty$

proof 2つのmapが互いに逆を与える //

Lem 4.1.17 [Boa, Lem 5.4]

(a) $\{F^sA^{-\infty}\}$ exhausts $A^{-\infty}$

(b)
- $\{F^sA^\infty\}$: complete Hausdorff
- $F^{-\infty}A^\infty = \operatorname{Ker}(A^\infty \to A^{-\infty})$

(In particular,

$A^{-\infty} = 0 \Rightarrow \{F^sA^\infty\}$ exhausts A^∞)

proof (a) 明らか (colim の basic property)

(b)

- $0 \to F^sA^\infty \to A^\infty \xrightarrow{\varepsilon^s} \operatorname{Im}^\sigma A^s \to 0$: exact

  $\downarrow \lim_s$

  $0 \to \lim_s F^sA^\infty \to A^\infty \xrightarrow[\cong]{} \lim_s \operatorname{Im}^\sigma A^s$ (Lem 4.1.16)

  $\to R\lim_s F^sA^\infty \to 0$

- $\operatorname{colim}_s F^sA^\infty = \operatorname{colim}_s \operatorname{Ker}(\varepsilon^s : A^\infty \to A^s)$

  $= \operatorname{Ker}(\operatorname{colim}_s \varepsilon^s : A^\infty \to \operatorname{colim}_s A^s)$ (colim is exact)

  $= \operatorname{Ker}(A^\infty \to A^{-\infty})$ //

Def 4.1.10 の意味で target にするために

$F^s/F^{s+1} \to E^s_\infty$ を def する.

Lem 4.1.18

(1) "$j \circ (\eta^s)^{-1}$": $F^sA^{-\infty}/F^{s+1}A^{-\infty} \to E^s_\infty$: well-def'd

$[x] \longmapsto [j(y)]$

(where

$x \in F^sA^{-\infty}$, $y \in A^s$ s.t. $\eta^s(y) = x$

$\rightsquigarrow j(y) \in \operatorname{Ker} k \subset Z^s_\infty$)

(2) Assume $A^{-\infty} = 0$

Then

"$k^{-1} \circ \varepsilon^{s+1}$": $F^sA^\infty/F^{s+1}A^\infty \to E^s_\infty$: well-def'd

$[x] \longmapsto [y]$

(where

$x \in F^sA^\infty$, $y \in Z^s_\infty \subset E^s$

s.t. $\varepsilon^{s+1}(x) = k(y)$ $(\in A^{s+1})$)

単に $y \in E^s$ で良い.

(∵ $y \in E^s$, $\varepsilon^{s+1}x = ky$

$\Rightarrow y \in k^{-1}(\operatorname{Im}\varepsilon^{s+1}) \subset k^{-1}(Q^{s+1}) = Z^s_\infty$)

証明の前に少し準備

Lem 4.1.18½

(1) $Z^s_\infty = k^{-1}(Q^{s+1})$

(2) $B^s_\infty = j(\operatorname{Ker}\eta^s)$

proof (1) $Z^s_\infty = \bigcap_r k^{-1}(\operatorname{Im}^{r-1}A^{s+1}) = k^{-1}(\bigcap_r \operatorname{Im}^{r-1}A^{s+1}) = k^{-1}(Q^{s+1})$

(2) $B^s_\infty = \bigcup_r j\operatorname{Ker}(A^s \to A^{s-r+1}) = j(\bigcup_r \operatorname{Ker}(A^s \to A^{s-r+1}))$

$= j(\operatorname{Ker}(A^s \to \bigcup_r A^{s-r+1})) = j(\operatorname{Ker}\eta^s)$ (colim: exact) //

proof of Lem 4.1.18

(1)

$$\begin{tikzcd}
A^s \arrow[r, "j"] \arrow[d, "\eta^s"] & \operatorname{Ker} k \arrow[r, hook] \arrow[d, phantom, "\circlearrowleft"] & Z^s_\infty \arrow[d] \\
F^sA^{-\infty} \arrow[rr, dashed, "\exists"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & & Z^s_\infty/B^s_\infty = E^s_\infty \\
F^sA^{-\infty}/F^{s+1}A^{-\infty} \arrow[urr, dashed, "\exists"'] & &
\end{tikzcd}$$

$(\subset E^s \supset)$

$B^s_\infty = j(\operatorname{Ker}\eta^s)$ by Lem 4.1.18½ (2)

$$\begin{tikzcd}
A^{s+1} \arrow[r, "i"] \arrow[d] \arrow[rr, bend right, "0"'] & A^s \arrow[r, "j"] & \operatorname{Ker} k \\
F^{s+1}A^{-\infty} & &
\end{tikzcd}$$

(∵)

(2)

$$\begin{tikzcd}
F^sA^\infty \arrow[r, "\varepsilon^{s+1}"] \arrow[d] \arrow[dr, dashed, "\exists"] & \overset{\operatorname{Im} k}{\overset{\|}{\operatorname{Ker} i}} \cap Q^{s+1} \\
F^sA^\infty/F^{s+1}A^\infty \arrow[r, dashed, "\exists"'] & Z^s_\infty/B^s_\infty \arrow[u, "k", "\cong"']
\end{tikzcd}$$

(∵ $F^{s+1}A^{-\infty} = \operatorname{Ker}\varepsilon^{s+1}$)

[∵ By Lem 4.1.18½,

$Z^s_\infty = k^{-1}(Q^{s+1})$

$B^s_\infty = j(\operatorname{Ker}(A^s \to \overset{\text{assume}}{\overset{0}{\overset{\|}{\boxed{A^{-\infty}}}}}))$

$= \operatorname{Im} j = \operatorname{Ker} k$]

//

以下で, $A^{\pm\infty}$ を target にするときは,

必ずこれらの map を用いる.

Lem4.1.19 [Boa, Lem5.6]

(1)
$$0 \to \frac{F^s A^{-\infty}}{F^{s+1}A^{\infty}} \longrightarrow E^s_\infty \xrightarrow{k} Q^{s+1} \xrightarrow{i} Q^s \to RE^s_\infty \to RQ^{s+1} \to RQ^s \to 0$$

"$j \circ (\eta^s)^{-1}$" : inj

exact

(2)
$$0 \to \frac{F^s A^{\infty}}{F^{s+1}A^{\infty}} \xrightarrow{\varepsilon^{s+1}} \mathrm{Im}^{\sigma} A^{s+1} \xrightarrow{i} \mathrm{Im}^{+} A^{s} \to 0 \quad : \text{exact}$$

(3) Assume $A^{-\infty} = 0$

Then We have a commutative diagram which relates (1) and (2) :

$$\begin{tikzcd}
0 \arrow[r] & E^s_\infty \arrow[r, "k"] & Q^{s+1} \arrow[r, "i"] & Q^s \arrow[r] & RE^s_\infty \arrow[r] & RQ^{s+1} \arrow[r] & RQ^s \arrow[r] & 0 \\
0 \arrow[r] & \frac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[u] \arrow[r, "\varepsilon^{s+1}"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[u, bend left, "\text{“}k^{-1}\circ\varepsilon^{s+1}\text{” : inj}"'] & \mathrm{Im}^\sigma A^{s+1} \arrow[u, hook] \arrow[r, "i"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Im}^\sigma A^s \arrow[u, hook] \arrow[r] & 0
\end{tikzcd}$$

exact

exact

少し準備

Lem 4.1.20

$N \overset{f}{\twoheadleftarrow} M \overset{f'}{\twoheadrightarrow} N'$: diagram of modules with f, f' : surj.

Then

$$\frac{N}{f(\mathrm{Ker} f')} \underset{f}{\overset{\cong}{\longleftarrow}} \frac{M}{\mathrm{Ker} f + \mathrm{Ker} f'} \underset{f'}{\overset{\cong}{\longrightarrow}} \frac{N'}{f'(\mathrm{Ker} f)} \quad : \text{isom}$$

proof surj は明らか. inj は元をとって計算するだけ //

Lem4.1.21

$$0 \to \frac{F^s A^{-\infty}}{F^{s+1}A^{-\infty}} \xrightarrow{\text{“}j\circ(\eta^s)^{-1}\text{”}} E^s_\infty \longrightarrow \frac{Z^s_\infty}{\mathrm{Ker}\,k} \to 0 \quad : \text{exact}$$

proof

$$0 \to \frac{\overset{\mathrm{Im}\,j\ \shortparallel}{\mathrm{Ker}\,k}}{B^s_\infty} \longrightarrow \overset{E^s_\infty\ \shortparallel}{\frac{Z^s_\infty}{B^s_\infty}} \longrightarrow \frac{Z^s_\infty}{\mathrm{Ker}\,k} \to 0 \quad : \text{exact} \quad \text{—— ①}$$

(both maps are induced by inclusions in E^s)

Apply Lem4.1.20 to the diagram

$$\mathrm{Im}\,j \underset{j}{\twoheadleftarrow} A^s \underset{\eta^s}{\twoheadrightarrow} F^s A^{-\infty}$$

$$\rightsquigarrow \frac{\mathrm{Im}\,j}{B^s_\infty} \underset{j}{\overset{\cong}{\longleftarrow}} \frac{A^s}{\mathrm{Ker}\,j + \mathrm{Ker}\,\eta^s} \underset{\eta^s}{\overset{\cong}{\longrightarrow}} \frac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \quad \text{—— ②}$$

($\because$ $B^s_\infty = j(\mathrm{Ker}\,\eta^s)$

$F^{s+1}A^{-\infty} = \mathrm{Im}(\eta^{s+1} : A^{s+1} \longrightarrow A^{-\infty}) = \eta^s(\mathrm{Im}\,i) = \eta^s(\mathrm{Ker}\,j)$)

$$\begin{tikzcd}
A^{s+1} \arrow[rr] \arrow[dr, "i"'] & & A^{-\infty} \\
& A^s \arrow[ur, "\eta^s"'] &
\end{tikzcd}$$

①② より ok //

proof of Lem4.1.19

(1)
$$0 \to \frac{Z^s_r}{\mathrm{Ker}\,k} \xrightarrow{k} \mathrm{Im}^{r-1}A^{s+1} \xrightarrow{i} \mathrm{Im}^{r}A^{s} \to 0 \quad : \text{exact}$$

($\because$ $Z^s_r = k^{-1}(\mathrm{Im}^{r-1}A^{s+1})$, $\mathrm{Ker}\,i = \mathrm{Im}\,k$)

$\lim_r$

$Z^s_\infty/\mathrm{Ker}\,k$ by Prop 3.4.3(b)

$\shortparallel$

$$0 \to \lim_r \frac{Z^s_r}{\mathrm{Ker}\,k} \xrightarrow{k} Q^{s+1} \xrightarrow{i} Q^s$$

$$\to R\lim_r \frac{Z^s_r}{\mathrm{Ker}\,k} \to RQ^{s+1} \to RQ^s \to 0 \quad : \text{exact}$$

$\shortparallel$ ← by Prop 3.4.3(c)

$R\lim_r Z^s_r$

$\shortparallel$

RE^s_∞

Splice this with the exact seq in Lem 4.1.21

(2)

$$\begin{tikzcd}
0 \arrow[r] & F^{s+1}A^\infty \arrow[r] \arrow[d, hook] & A^\infty \arrow[r, "\varepsilon^{s+1}"] \arrow[d, equal] & \mathrm{Im}^\sigma A^{s+1} \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & F^sA^\infty \arrow[r] & A^\infty \arrow[r, "\varepsilon^s"] & \mathrm{Im}^\sigma A^s \arrow[r] & 0
\end{tikzcd}$$

: exact

: exact

snake lemma

$$\mathrm{Ker}(i : \mathrm{Im}^\sigma A^{s+1} \to \mathrm{Im}^\sigma A^s) \underset{(\varepsilon^{s+1})^{-1}}{\overset{\cong}{\longrightarrow}} \frac{F^sA^\infty}{F^{s+1}A^\infty} \quad : \text{isom}$$

$\overset{\cong}{\longleftarrow}$ ε^{s+1}

(3) 明らか. //

Rmk4.1.22

Lem4.1.19(3) を $k^{-1}\circ\varepsilon^{s+1}$ の def としても良い

$$\begin{tikzcd}
0 \arrow[r] & E^s_\infty \arrow[r] & Q^{s+1} \arrow[r] & Q^s \arrow[r] & \cdots \\
0 \arrow[r] & \frac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[u, dashed, "\exists !"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Im}^\sigma A^{s+1} \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Im}^\sigma A^s \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

$A^{-\infty} = 0$

: exact

: exact

(ただし, $k^{-1}\circ\varepsilon^{s+1}$ は重要な map なので Lem4.1.18 で直接 def を与えた)

$RE_\infty = 0 \Rightarrow {}^\exists$ simplification of Lem 4.1.19

Lem 4.1.23 [Boa, Lem 5.9]

Assume

$RE_\infty = 0$

Then

(a) $\forall s$, $\varepsilon^s : A^\infty \longrightarrow Q^s$: surj

($\underline{\text{i.e}}$ $\mathrm{Im}^\infty A^s = Q^s$)

(b) $\forall s$, $RA^\infty \xrightarrow{\cong} RQ^s$: isom

$$\left(\begin{array}{ccc} \| & & \| \\ R\lim_r A^{r+s} & \longrightarrow & R\lim_r \mathrm{Im}^r A^s \end{array} \right)$$

(c) TFAE

(1) $\{E_r\}$ converges weakly to $A^{-\infty}$

(2) $\forall s$, $\varepsilon^s : A^\infty \longrightarrow A^s$: inj

(3) $\forall s$, $\varepsilon^s : A^\infty \xrightarrow{\cong} Q^s$: isom

proof

Since $RE_\infty = 0$, Lem 4.1.19 (1) breaks up into :

$$\begin{cases} 0 \to \dfrac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \to E_\infty^s \longrightarrow Q^{s+1} \xrightarrow{i} Q^s \to 0 \ \text{: exact} & \text{—— ①} \\ RQ^{s+1} \xrightarrow{\cong} RQ^s & \text{—— ②} \end{cases}$$

(a) By ①, $Q^{s+1} \twoheadrightarrow Q^s$: surj

$\xrightarrow{\text{Prop3.2.5(a)}}$

$$\begin{tikzcd} \lim_s Q^s \arrow[r, two heads] \arrow[d, "\cong"', "\text{Thm3.5.3(a)}"] \arrow[rd, phantom, "\circlearrowleft"] & Q^s \ \text{: surj} \\ A^\infty \arrow[ru, "\varepsilon^s"'] & \end{tikzcd}$$

(b) By ②, $\forall s$, $\lim_t RQ^t \xrightarrow{\cong} RQ^s$: isom

By Thm 3.5.3 (b) (ML exact seq)

$$\begin{tikzcd} 0 \arrow[r] & \cancel{R\lim_t Q^t} \arrow[r] & RA^\infty \arrow[r] \arrow[rd] & \lim_t RQ^t \arrow[r] \arrow[d, "\cong"] \arrow[ld, phantom, "\circlearrowleft"] & 0 \ \text{: exact} \\ & & & RQ^s & \end{tikzcd}$$

$\cancel{R\lim_t Q^t} = 0$

(∵ $Q^{s+1} \to Q^s$: surj
Prop3.2.5 (b))

∵
$$\begin{tikzcd} A^{t+r} \arrow[r] \arrow[rd] \arrow[r, phantom, shift right=3ex, "\circlearrowleft"] & \mathrm{Im}^r A^t \arrow[d] \\ & \mathrm{Im}^r A^s \end{tikzcd}$$
$(t \geq s)$

$R\lim_r$ {

$$\begin{tikzcd} RA^\infty \arrow[r] \arrow[rd] \arrow[r, phantom, shift right=3ex, "\circlearrowleft"] & RQ^t \arrow[d] \\ & RQ^s \end{tikzcd}$$

Cor3.5.15

$\xrightarrow[\lim_t]{}$ 上図

(c) $\underline{(2) \Leftrightarrow (3)}$ (a) より ok

$\underline{(1) \Leftrightarrow (3)}$ $(1) \underset{\text{① Lem4.1.17(a)}}{\Longleftrightarrow} \forall s,\ Q^{s+1} \underset{\text{isom}}{\xrightarrow{\cong}} Q^s \Longleftrightarrow (3)$

//

Conditional convergence

Def 4.1.24 [Boa, Def 5.10]

- $\{E_r\}$ $\underline{\text{converges conditionally to the colimit } A^\infty}$

$\underset{\text{def}}{\Longleftrightarrow}$ $A^\infty = RA^\infty = 0$

- $\{E_r\}$ $\underline{\text{converges conditionally to the limit } A^{\infty}}$

$\underset{\text{def}}{\Longleftrightarrow}$ $A^{-\infty} = 0$

Rmk 4.1.25

- 一般には、

conditionally conv. $\not\Rightarrow$ weakly conv. (see Ex5.1.18)

- しかし、E_rの有限性などの追加の仮定があれば

strongly conv. まで言える。

conditionally conv. $\Rightarrow {}^\exists$ simplification of Lem 4.1.19

Lem 4.1.26 [Boa, Lem 5.11]

Assume

$\{E_r\}$ conditionally converges to the colim $A^{-\infty}$

Then ($A^\infty = 0$ は不要)

(a) $\forall s$, $RQ^s = 0$

(b) $\{F^s A^{-\infty}\}$: complete ($\underline{\text{i.e.}}$ $R\lim_s F^s A^{-\infty} = 0$)

(c)
$$0 \to \frac{F^s A^{\infty}}{F^{s+1} A^{\infty}} \to E_\infty^s \to Q^{s+1} \to Q^s \to RE_\infty^s \to 0$$
exact

proof

(a) $RA^\infty = 0 \xrightarrow{\text{Cor3.5.5}} \forall s,\ RQ^s = 0$

(c) follows from Lem 4.1.19 (1) and (a)

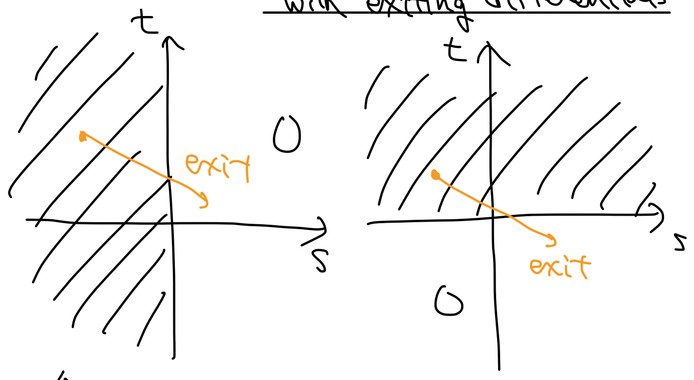
(b) $\eta^s : A^s \twoheadrightarrow F^s A^{-\infty}$: surj (by def)

{$R\lim_s$: right exact

$$\begin{array}{l} RA^\infty \twoheadrightarrow R\lim_s F^s A^{-\infty} \ \text{: surj} \\ \| \\ 0 \end{array}$$

//

§4.2 Half-plane spectral sequences with exiting differentials



← 詳細を扱うのはこの case だけだが,
一般の case も同様にできる
degree-wise

Thm 4.2.1 [Boa, Thm 6.1]

Assume

$\forall s > 0,\ E^s = 0$

Then

(a) If $A^\infty = 0$, then

$\{E_r\}$ converges strongly to $A^{-\infty}$

by the map $j \circ (\eta^s)^{-1}$ in Lem 4.1.18

(b) If $A^{-\infty} = 0$, then

$\{E_r\}$ converges strongly to A^∞

by the map $k^{-1} \circ \varepsilon^{s+1}$ in Lem 4.1.18

一般の case も statement だけ書いておく

Thm 4.2.1'

Consider the case

$\deg(i) = 0,\ \deg(j) = 0,\ \deg(k) = +1$

$\left(\rightsquigarrow d_r^{st} : E_r^{st} \longrightarrow E_r^{s+r,\, t-r+1}\right)$

Assume

$\forall n,\ \exists s_0(n),\ \forall s > s_0(n)\ E^{s,n-s} = 0$

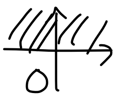
Then

Same (a) (b) as in Thm 4.2.1

← e.g.

(1) $s_0(n) = 0 \rightsquigarrow$ Thm 4.2.1

(2) $s_0(n) = n \rightsquigarrow$



Lem 4.2.2

TFAE:

(1) $\forall s > 0,\ E^s = 0$

(2) $\forall s > 0,\ \varepsilon^s : A^\infty \xrightarrow{\cong} A^s$: isom

(3) $\forall s > 0,\ i : A^{s+1} \xrightarrow{\cong} A^s$: isom

proof

$$\begin{tikzcd}
{} \arrow[r, "\cong"] & A^2 \arrow[rr, "\cong"] \arrow[dl] & & A^1 \arrow[rr] \arrow[dl] & & A^0 \arrow[rr] \arrow[dl] & & A^{-1} \arrow[r] \arrow[dl] & {} \\
{} & & \underset{0}{\cancel{E^1}} \arrow[ul] & & E^0 \arrow[ul] & & E^{-1} \arrow[ul] & &
\end{tikzcd}$$

//

proof of Thm 4.2.1

(a)

exhausts 一般に成立 (Lem 4.1.17 (a))

complete Hausdorff

$\forall s > 0$ に対し, $F^s A^{-\infty} = \mathrm{Im}(\eta^s : \overset{\overset{A^\infty = 0}{\shortparallel}}{A^s} \to A^{-\infty}) = 0$

$\rightsquigarrow \lim_s F^s A^{-\infty} = R\lim_s F^s A^{\infty} = 0$

isom

$\forall s,\ Q^s = 0$

$\left(∵\ A^1 = 0 \rightsquigarrow \mathrm{Im}^r A^s = 0 \text{ for } r > 1-s \right)$

$\xrightarrow[\text{Lem 4.1.19 (1)}]{\rightsquigarrow} j \circ (\eta^s)^{-1}$: isom

(b)

exhausts

Lem 4.1.17 (b) より,

$\operatorname{colim}_s F^s A^\infty = \mathrm{Ker}(A^\infty \longrightarrow \overset{\overset{0}{\shortparallel}}{A^{-\infty}}) = A^\infty$

complete Hausdorff

$\forall s > 0$ に対し, $F^s A^\infty = \mathrm{Ker}(\varepsilon^s : A^\infty \xrightarrow{\cong} A^s) = 0$

$\rightsquigarrow \lim_s F^s A^\infty = R\lim_s F^s A^\infty = 0$

isom

$\forall s,\ \mathrm{Im}^\infty A^s = Q^s$

$\left(∵\ \underline{s \geq 1}\ \ \mathrm{Im}^\infty A^s = Q^s = A^s \right.$

$\left. \underline{s \leq 1}\ \ \mathrm{Im}^r A^s \underset{\substack{\uparrow \\ \text{for } r > 1-s}}{=} \mathrm{Im}(\underset{\underset{A^\infty}{\wr}}{A^1} \longrightarrow A^s) = \mathrm{Im}^\infty A^s \right)$

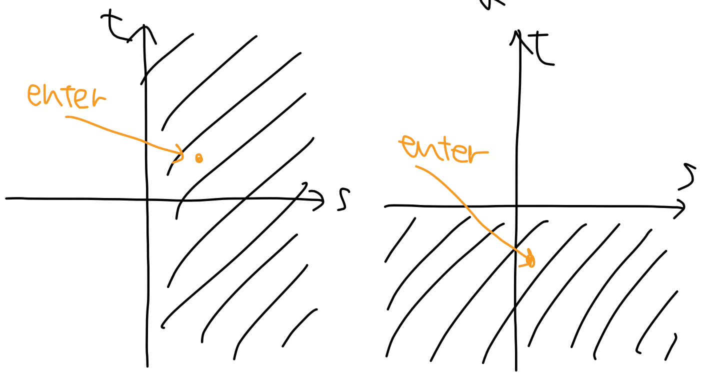
$\xrightarrow[\text{Lem 4.1.19 (3) + 5-lemma}]{\rightsquigarrow} k^{-1} \circ \varepsilon^{s+1}$: isom

//

Rmk 4.2.3

Comparison thm は Thm 4.1.13 で十分

§4.3 Half-plane spectral sequences with entering differentials



↖ 詳細を扱うのはこのcaseだけだが、
一般のcaseも同様にできる
degree-wise

Thm4.3.1 [Boa, Thm7.1]

Assume

- $\forall s<0,\ E^s=0$
- $\{E_r\}$ converges conditionally to $A^{-\infty}$ (resp. A^{∞})

Then

$RE_\infty=0$

$\Rightarrow$ $\{E_r\}$ converges strongly to $A^{-\infty}$ by $j\circ(\eta^s)^{-1}$
(resp. A^{∞} by $k^{-1}\circ\varepsilon^{s+1}$)

(see Thm4.3.5 and Thm4.3.6 for the proof)

一般のcaseもstatementだけ書いておく

Thm4.3.1′

Consider the case

$\deg(i)=0,\ \deg(j)=0,\ \deg(k)=+1$

$\left(\rightsquigarrow d_r^{st}: E_r^{st}\longrightarrow E_r^{s+r,\,t-r+1}\right)$

Assume

- $\forall n,\ \exists s_0(n),\ \forall s<s_0(n)\ E^{s,n-s}=0$
- $\{E_r\}$ converges conditionally to $A^{-\infty}$ (resp. A^{∞})

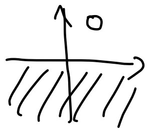
Then

Same as in Thm4.3.1

eg.

(1) $s_0(n)=0 \rightsquigarrow$ Thm4.3.1

(2) $s_0(n)=n \rightsquigarrow$



Rmk4.3.2

- $RE_\infty=0$ となる十分条件については、Prop4.1.4, Cor4.1.8 を参照。
- Thm4.3.1により、strong convergenceは以下の2つの問題に分割される：
  - conditional convergence：
    - structural condition
    - holds for large classes of s.s.
  - $RE_\infty=0$：
    - depends only on the data internal to s.s.,
    - cannot be expected to hold in general

証明の前に少し準備

Lem4.3.3

TFAE：

(1) $\forall s<0,\ E^s=0$

(2) $\forall s\le 0,\ \eta^s: A^s\xrightarrow{\simeq}A^{-\infty}$: isom

(3) $\forall s\le 0,\ i: A^s\xrightarrow{\simeq}A^{s-1}$: isom

proof

$$\begin{tikzcd}
{} \arrow[r] & A^1 \arrow[r] & A^0 \arrow[r, "\simeq"] \arrow[dl] & A^{-1} \arrow[r, "\simeq"] \arrow[dl] & A^{-2} \arrow[r] \arrow[dl] & \cdots \\
 & E^0 \arrow[ul] & \cancel{E^{-1}} \arrow[ul] & \cancel{E^{-2}} \arrow[ul] & &
\end{tikzcd}$$

//

Lem4.3.4

Assume

$\forall s<0,\ E^s=0$

Then

For $\forall s\le 0$, $\eta^s: A^s\xrightarrow{\simeq}A^{-\infty}$ induces

(1) $Q^s\xrightarrow{\simeq}F^\infty A^{-\infty}$ $\left(:=\lim_r F^rA^{-\infty}\right)$

(2) $RQ^s\xrightarrow{\simeq}RF^\infty A^{-\infty}$ $\left(:=R\lim_r F^rA^{-\infty}\right)$

proof

$$\begin{tikzcd}
\eta^s: A^s \arrow[r, "\simeq"] \arrow[d, phantom, "\cup"] & A^{-\infty} \arrow[d, phantom, "\vee"] \\
\mathrm{Im}^r A^s \arrow[r, "\simeq"] & F^{s+r}A^{-\infty}
\end{tikzcd}$$

: isom of filtered module

//

Thm 4.3.1は，colim と lim 別々に証明する

◉ colim as target

より強く，次が言える:

Thm 4.3.5 [Boa, Thm 7.3]

> **Assume**
>
> $\forall s<0,\ E^s=0$
>
> **Then**
>
> 2 of the following ⇒ 3rd:
>
> (1) converge conditionally to $A^{-\infty}$
>
> (i.e. $A^\infty = RA^\infty = 0$)
>
> (2) $RE_\infty = 0$
>
> (3) converge strongly to $A^{-\infty}$ by $j\circ(\eta^s)^{-1}$

proof

$\{F^sA^{-\infty}\}_s$: exhaustive

(∵ Lem 4.1.17 (a) より，一般に成立

(より強く，次も言える:

$F^0A^{-\infty} = \mathrm{Im}(\eta^0 : A^0 \xrightarrow{\cong} A^{-\infty}) = A^{-\infty}$))

(1)(2) ⇒ (3) (これが Thm 4.3.1 for colim)

We need to prove:

- $F^\infty A^{-\infty} = 0$ (Hausdorff)
- $RF^\infty A^{-\infty} = 0$ (complete)
- $j\circ(\eta^s)^{-1} : \dfrac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \xrightarrow{\cong} E^s_\infty$: isom

- $F^\infty A^{-\infty} \cong Q^0 = \mathrm{Im}^\sigma A^0 = 0$

  ↑ Lem 4.3.4(1)　↑ Lem 4.1.23 (a)　↑ (1) $A^\infty = 0$

  ↖ (2) $RE_\infty = 0$

- $RF^\infty A^{-\infty} \cong RQ^0 \cong RA^\infty = 0$

  ↑ Lem 4.3.4(2)　↑ Lem 4.1.23 (b)　↖ (1) $RA^\infty = 0$

  (2) $RE_\infty = 0$　↖ (2) $RE_\infty = 0$

- By Lem 4.1.23 (c) (2) ⇒ (1),

  ↖ (1) $A^\infty = 0$

  $\{E_r\}$: weakly convergent

  ↪ $\eta\circ(\varepsilon^s)^{-1}$: isom

(2)(3) ⇒ (1)

- $A^\infty \cong Q^0 \cong F^\infty A^{-\infty} = 0$

  Lem 4.3.4(1)

  ↑ Lem 4.1.23 (c) (1) ⇒ (3)　↖ (3) Hausdorff

  ↑ (2) $RE_\infty = 0$　↖ (3) weakly conv.

- $RA^\infty \cong RQ^0 \cong RF^\infty A^{-\infty} = 0$

  ↑ Lem 4.1.23 (b)　↖ Lem 4.3.4(2)　↖ (3) complete

  ↖ (2) $RE_\infty = 0$

(1) (3) ⇒ (2)

$\forall s,\ Q^s = 0$

(∵ $s \le 0$

Lem 4.3.4(1)　(3) Hausdorff

$Q^s \cong F^\infty A^{-\infty} = 0$

$s \ge 0$

By Lem 4.1.26 (c) ← (1) $RA^\infty = 0$

$$0 \to \frac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \xrightarrow{\cong} E^s_\infty \to Q^{s+1} \to Q^s \to RE^s_\infty \to 0$$

↖ (3) $j\circ(\eta^s)^{-1}$: isom

↪ $\forall s,\ Q^{s+1} \to Q^s$: inj.

Since $Q^0 = 0$, this proves the claim.)

$\xrightarrow[\text{Lem 4.1.26(c)}]{}$ $\forall s.\ RE^s_\infty = 0$

$$\left(0 \to \frac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \xrightarrow{\cong} E^s_\infty \to \underset{0}{\cancel{Q^{s+1}}} \to \underset{0}{\cancel{Q^s}} \to RE^s_\infty \to 0 \right)$$

//

lim as target

より強く、次が示せる：

Thm4.3.6 [Boa, Thm 7.4]

Assume

$\forall s \leq 0,\ A^s = 0$

(i.e. ・$\forall s < 0,\ E^s = 0$
・conditionally convergent to A^∞
($\underline{\text{i.e.}}$ $A^{-\infty} = 0$))

Then TFAE:

(1) $RE_\infty = 0$

(2) $RA^\infty = 0$ and
converges strongly to A^∞ by $k^{-1}\circ\varepsilon^{s+1}$

proof

・Since $A^{-\infty} = 0$, by Lem4.1.19(3), we have :

$$\begin{tikzcd}
0 \arrow[r] & E^s_\infty \arrow[r, "k"] & Q^{s+1} \arrow[r, "i"] & Q^s \arrow[r] & RE^s_\infty \arrow[r] & RQ^{s+1} \arrow[r] & RQ^s \arrow[r] & 0 \\
 & & & & & & \text{exact} & \\
0 \arrow[r] & \frac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[uu, "k^{-1}\circ\varepsilon^{s+1}"] \arrow[r, "\varepsilon^{s+1}"] \arrow[uur, phantom, "\circlearrowleft"] & \mathrm{Im}^\infty A^{s+1} \arrow[uu, hook] \arrow[r, "i"] \arrow[uur, phantom, "\circlearrowleft"] & \mathrm{Im}^\infty A^s \arrow[uu, hook] \arrow[r] & 0 & \text{exact} & &
\end{tikzcd}$$

— ①

・$\forall s \leq 0,\ Q^s = RQ^s = 0$ — ②

(∵ $A^s = 0 \rightsquigarrow \mathrm{Im}^r A^s = 0$ for $s \leq 0$)

$\underline{(1) \Rightarrow (2)}$ (これが Thm 4.3.1 for lim)

(1) $RE_\infty = 0$

$\xrightarrow[\text{Lem4.1.23(a)(b)}]{}$ $\begin{cases} \cdot\ RA^\infty \cong RQ^0 \overset{②}{=} 0 \\ \cdot\ \forall s,\ \mathrm{Im}^\infty A^s \cong Q^s \quad \text{— ③} \end{cases}$

①③ + 5-lemma $\rightsquigarrow$ $k^{-1}\circ\varepsilon^{s+1}$: isom

Lem 4.1.17より,

$\{F^sA^\infty\}_s$ は $\begin{cases} \text{常に complete Hausdorff} \\ \text{exhaustive } (∵ A^{-\infty} = 0) \end{cases}$

$\rightsquigarrow$ strongly convergent

$\underline{(2) \Rightarrow (1)}$

$\forall s,\ \mathrm{Im}^\infty A^s = Q^s$

(∵ $\underline{s \leq 0}$ $A^s = 0$ なので明らか.
$\underline{s \geq 0}$ ① で 5-lemma を使った induction) ← (2) $k^{-1}\circ\varepsilon^{s+1}$: isom

Hence, by ①, $\forall s.\ Q^{s+1} \twoheadrightarrow Q^s$: surj.

ところで,

(2) $RA^\infty = 0 \xrightarrow{\text{Cor3.5.5}} \forall s.\ RQ^s = 0$

$\Big\} \xrightarrow{①} \forall s.\ RE^s_\infty = 0$

//

以上で Thm4.3.1 が示せた.

Comparison theorem

strongly conv. とは限らないので, Thm4.1.13 では不十分.

Thm4.3.7 [Boa, Thm 7.2]

$f : (A^s, E^s) \longrightarrow (\bar{A}^s, \bar{E}^s)$: morph of unrolled exact couple

Assume

・$\forall s < 0,\ E^s = \bar{E}^s = 0$

・$\{E_r\}$ and $\{\bar{E}_r\}$ conditionally converges to either (1) or (2) :

$\begin{cases} (1)\ G = A^{-\infty} \text{ and } \underline{\bar{G} = \bar{A}^{-\infty}} \\ (2)\ G = A^\infty \text{ and } \bar{G} = \bar{A}^\infty \end{cases}$ ← $R\bar{A}^\infty = 0$ は不要

・$f_\infty : E_\infty \xrightarrow{\cong} \bar{E}_\infty$: isom
$RE_\infty \xrightarrow{\cong} R\bar{E}_\infty$: isom

Then

(a) $\forall s.\ Q^s \xrightarrow{\cong} \bar{Q}^s$: isom
(b) $\forall s.\ RQ^s \xrightarrow{\cong} R\bar{Q}^s$: isom

証明を整理するために追加した.

(c) $G \xrightarrow{\cong} \bar{G}$: isom of filtered modules
($\underline{\text{i.e.}}$ ・$G \xrightarrow{\cong} \bar{G}$ as modules
・$\forall s.\ F^sG \xrightarrow{\cong} F^s\bar{G}$)

(d) $RA^\infty \xrightarrow{\cong} R\bar{A}^\infty$

Rmk4.3.8

・(1) の case では, (d) から $R\bar{A}^\infty = 0$ が従う.
↖仮定ではない.

・$RE_\infty = R\bar{E}_\infty = 0$ を仮定していないので.
$\{E_r\}, \{\bar{E}_r\}$ は weakly convergent ですらない
$\rightsquigarrow$ conditional convergence は, 理論だけでなく practical computations にも使える tool.
(解析で divergent series を考えるのが有用である のと同様に, (Def4.1.11 の意味で) 収束しない s.s. を考えるのが有用となる.)

↑
例ってあるの??

proof of Thm 4.3.7 in the case (i)

($RA^\infty = 0$ は仮定しない) $RA^\infty = 0$

Lem 4.1.26 (for (A^s, E^s)), Lem 4.1.19 (for $(\overline{A}^s, \overline{E}^s)$) より.

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & K^{s+1} \arrow[r] \arrow[d] & K^s \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \leftarrow \mathrm{Ker} & \\
0 \arrow[r] & \frac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & E^s_\infty \arrow[r] \arrow[d, "\cong"'] \arrow[rd, phantom, "\circlearrowleft"] & Q^{s+1} \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & Q^s \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & RE^s_\infty \arrow[r] \arrow[d, "\cong"'] \arrow[rd, phantom, "\circlearrowleft"] & 0 \arrow[d] & \text{exact} & \text{— ①} \\
0 \arrow[r] & \frac{F^s\overline{A}^{-\infty}}{F^{s+1}\overline{A}^{-\infty}} \arrow[r] & \overline{E}^s_\infty \arrow[r] \arrow[d] & \overline{Q}^{s+1} \arrow[r] \arrow[d] & \overline{Q}^s \arrow[r] \arrow[d] & R\overline{E}^s_\infty \arrow[r] \arrow[d] & R\overline{Q}^{s+1} \arrow[r] & R\overline{Q}^s \arrow[r] & 0 \\
 & & 0 \arrow[r] & C^{s+1} \arrow[r] & C^s \arrow[r] & 0 & & \leftarrow \mathrm{Coker} & \text{exact}
\end{tikzcd}$$

(a) $Q^s \xrightarrow{\cong} \overline{Q}^s$

$K^s := \mathrm{Ker}(Q^s \longrightarrow \overline{Q}^s)$, $C^s := \mathrm{Coker}(Q^s \to \overline{Q}^s)$ とおく.

① に Lem 3.2.15 を用いて.

$$\begin{cases} \forall s,\ K^{s+1} \twoheadrightarrow K^s \ \text{, surj} & \text{—— ②} \\ \forall s,\ C^{s+1} \xrightarrow{\cong} C^s \ \text{: isom} & \text{—— ③} \end{cases}$$

よって

$\forall s,\ K^s = 0$

(∵ $0 \to K^s \to Q^s \to \overline{Q}^s$: exact

$\xrightarrow{\lim_s}$ $0 \to \lim_s K^s \to \lim_s Q^s \to \lim_s \overline{Q}^s$: exact

$$\begin{tikzcd}
\lim_s Q^s \arrow[d, "\cong", "\text{Thm3.5.3(a)}"'] \\
\lim_s A^s = 0 \quad \leftarrow \text{assump. } A^\infty = 0
\end{tikzcd}$$

$\rightsquigarrow \lim_s K^s = 0$

よって ② と Cor 3.2.6 より $\forall s,\ K^s = 0$)

また,

$\forall s,\ C^s = 0$

(∵ $K^s = 0$ より.

$0 \to Q^s \to \overline{Q}^s \to C^s \to 0$: exact

$\downarrow \lim_s$

$0 \to \lim_s Q^s \to \boxed{\lim_s \overline{Q}^s} \to \lim_s C^s$ $\quad$ ($\lim_s \overline{Q}^s = A^\infty = 0$ ← assump)

$\to \boxed{R\lim_s Q^s} \to R\lim_s \overline{Q}^s \to R\lim_s C^s \to 0$: exact

$R\lim_s Q^s = 0$ ← $RA^\infty = 0$, Thm 3.5.3(b) (ML exact seq.)

$\rightsquigarrow \lim_s C^s = 0$

$\overset{③}{\rightsquigarrow}$ $\forall s,\ C^s = 0$)

以上より.

$\forall s,\ Q^s \xrightarrow{\cong} \overline{Q}^s$: isom

(c) $A^{-\infty} \xrightarrow{\cong} \overline{A}^{-\infty}$

$Q^s \longrightarrow \overline{Q}^s$: inj で十分 (surj は下で使う)

① と (a) より.

$$\forall s,\quad \frac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \xrightarrow{\cong} \frac{F^s\overline{A}^{-\infty}}{F^{s+1}\overline{A}^{-\infty}}$$

よって Thm 3.4.6 より.

$A^{-\infty} \xrightarrow{\cong} \overline{A}^{-\infty}$: isom of filtered modules

(∵ Thm 3.4.6 の仮定を check する:

$\{F^sA^{-\infty}\}$, $\{F^s\overline{A}^{-\infty}\}$: exhaustive

Lem 4.1.17 (a) より常に成立

$F^\infty A^{-\infty} \xrightarrow{\cong} F^\infty\overline{A}^{-\infty}$

Lem 4.3.4 (1) より.

$$\begin{tikzcd}
Q^0 \arrow[r, "\cong"] \arrow[d, "(a)\ \cong"'] \arrow[rd, phantom, "\circlearrowleft"] & F^\infty A^{-\infty} \arrow[d] \\
\overline{Q}^0 \arrow[r, "\cong"] & F^\infty\overline{A}^{-\infty}
\end{tikzcd}$$

$\{F^sA^{-\infty}\}$: complete ← $\{F^s\overline{A}^{-\infty}\}$ については不要

$RA^\infty = 0$ なので Lem 4.1.26 (b) より ok.)
← assump

(b) $RQ^s \xrightarrow{\cong} R\overline{Q}^s$

$RA^\infty = 0$, Cor 3.5.5 より

$\forall s,\ RQ^s = 0$

($\rightsquigarrow$ 「$\forall s,\ R\overline{Q}^s = 0$」 を示せば良い.)

ここで Lem 4.3.4 (2) より.

$$\begin{tikzcd}
0 = RQ^0 \arrow[r, "\cong"] \arrow[d] & RF^\infty A^\infty \arrow[d, "\cong \leftarrow (c)"] \\
R\overline{Q}^0 \arrow[r, "\cong"] & RF^\infty\overline{A}^{-\infty}
\end{tikzcd}$$

$\rightsquigarrow R\overline{Q}^0 = 0$

また ① より

$\forall s,\ R\overline{Q}^{s+1} \xrightarrow{\cong} R\overline{Q}^s$: isom

$$\left(∵ \begin{tikzcd}
Q^s \arrow[r, "\text{surj}"'] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & RE^s_\infty \arrow[r] \arrow[d] & 0 & & \\
\overline{Q}^s \arrow[r, "\text{surj}"'] \arrow[rr, bend right, "0"'] & R\overline{E}^s_\infty \arrow[r] & R\overline{Q}^{s+1} \arrow[r, "\text{inj}"'] & R\overline{Q}^s \arrow[r] & 0
\end{tikzcd} \right)$$

以上より

$\forall s,\ R\overline{Q}^s = 0$

(d) $RA^\infty \xrightarrow{\cong} R\overline{A}^\infty$

(a) (b) と Thm 3.5.3 (b) (ML exact seq) より ok

$$\begin{tikzcd}
0 \arrow[r] & R\lim_s Q^s \arrow[r] \arrow[d, "(a)\ \cong"'] \arrow[rd, phantom, "\circlearrowleft"] & RA^\infty \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & \lim_s RQ^s \arrow[r] \arrow[d, "\cong (b)"] & 0 \\
0 \arrow[r] & R\lim_s \overline{Q}^s \arrow[r] & R\overline{A}^\infty \arrow[r] & \lim_s R\overline{Q}^s \arrow[r] & 0
\end{tikzcd}$$

(実際には全て 0 だけど)

//

proof of Thm 4.3.7 in the case (2)

Lem 4.1.19 (1) より

$$\begin{tikzcd}
A^{-\infty}=0 \arrow[d] & 0 \arrow[r] \arrow[d] & K^{s+1} \arrow[r] \arrow[d] & K^s \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & L^{s+1} \arrow[r] \arrow[d] & L^s \arrow[r] \arrow[d] & 0 \arrow[d] & \mathrm{Ker} \arrow[l] \\
0 \arrow[r] & E^s_\infty \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & Q^{s+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Q^s \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & RE^s_\infty \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & RQ^{s+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & RQ^s \arrow[r] \arrow[d] & 0 \arrow[d] & :\text{exact} \arrow[d, phantom, "\text{— ④}"] \\
0 \arrow[r] & \overline{E}^s_\infty \arrow[r] \arrow[d] & \overline{Q}^{s+1} \arrow[r] \arrow[d] & \overline{Q}^s \arrow[r] \arrow[d] & R\overline{E}^s_\infty \arrow[r] \arrow[d] & R\overline{Q}^{s+1} \arrow[r] \arrow[d] & R\overline{Q}^s \arrow[r] \arrow[d] & 0 \arrow[d] & :\text{exact} \\
\overline{A}^{-\infty}=0 \arrow[u] & 0 \arrow[r] & C^{s+1} \arrow[r] & C^s \arrow[r] & 0 \arrow[r] & D^{s+1} \arrow[r] & D^s \arrow[r] & 0 & \mathrm{Coker} \arrow[l]
\end{tikzcd}$$

(a) $\underline{Q^s \xrightarrow{\cong} \overline{Q}^s}$

$K^s := \mathrm{Ker}(Q^s \longrightarrow \overline{Q}^s)$, $C^s := \mathrm{Coker}(Q^s \to \overline{Q}^s)$ とおいて.

④ と Lem 3.2.15 より.

$$\begin{cases} \forall s.\ K^{s+1} \xrightarrow{\cong} K^s : \text{isom} \\ \forall s.\ C^{s+1} \rightarrowtail C^s : \text{inj} \end{cases} \quad \text{— ⑤}$$

ここで

$\forall s \le 0,\ A^s \cong A^{-\infty} \overset{\text{assump}}{=} 0,\ \overline{A}^s \cong \overline{A}^{-\infty} = 0$

$\rightsquigarrow \forall s \le 0,\ Q^s = \overline{Q}^s = 0 \quad (\because Q^s \subset A^s)$

$\rightsquigarrow \forall s \le 0,\ K^s = C^s = 0$

$\underset{⑤}{\rightsquigarrow} \forall s,\ K^s = C^s = 0$

(b) $\underline{RQ^s \xrightarrow{\cong} R\overline{Q}^s}$

$L^s := \mathrm{Ker}(RQ^s \to R\overline{Q}^s)$, $D^s := \mathrm{Coker}(RQ^s \to R\overline{Q}^s)$

(a), ④, Lem 3.2.15 より.

$$\begin{cases} \forall s.\ L^{s+1} \xrightarrow{\cong} L^s : \text{isom} \\ \forall s.\ D^{s+1} \xrightarrow{\cong} D^s : \text{isom} \end{cases} \quad \text{— ⑥}$$

ここで

$A^0 = \overline{A}^0 = 0$

$\rightsquigarrow \forall r,\ \mathrm{Im}^r A^0 = \mathrm{Im}^r \overline{A}^0 = 0$

$\rightsquigarrow RQ^0 = R\overline{Q}^0 = 0 \quad (\because RQ^0 = R\lim_r \mathrm{Im}^r A^0)$

$\rightsquigarrow L^0 = D^0 = 0$

$\underset{⑥}{\rightsquigarrow} \forall s,\ L^s = D^s = 0$

(c) $\underline{A^\infty \xrightarrow{\cong} \overline{A}^\infty}$

(a) と Cor 3.5.19 より.

$\forall s,\ \mathrm{Im}^a A^s \xrightarrow{\cong} \mathrm{Im}^a \overline{A}^s$

Lem 4.1.19 (2) より

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Im}^a A^{s+1} \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Im}^a A^s \arrow[r] \arrow[d, "\cong"] & 0 & :\text{exact} \\
0 \arrow[r] & \dfrac{F^s\overline{A}^\infty}{F^{s+1}\overline{A}^\infty} \arrow[r] & \mathrm{Im}^a \overline{A}^{s+1} \arrow[r] & \mathrm{Im}^a \overline{A}^s \arrow[r] & 0 & :\text{exact}
\end{tikzcd}$$

よって

$$\forall s,\ \frac{F^sA^\infty}{F^{s+1}A^\infty} \xrightarrow{\cong} \frac{F^s\overline{A}^\infty}{F^{s+1}\overline{A}^\infty} : \text{isom}$$

また. Lem 4.1.17 (b) より.

$\{F^sA^\infty\}, \{F^s\overline{A}^\infty\}$: complete, Hausdorff, $\underset{\uparrow\, A^{-\infty} = \overline{A}^{-\infty} = 0}{\underline{\text{exhaustive}}}$

よって Thm 3.4.6 より.

$A^\infty \xrightarrow{\cong} \overline{A}^\infty$: isom of filtered modules

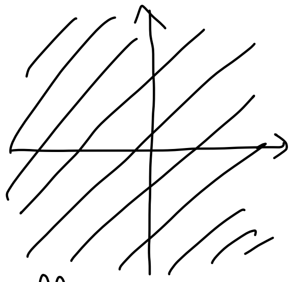
(d) $\underline{RA^\infty \xrightarrow{\cong} R\overline{A}^\infty}$

(a) (b) と Thm 3.5.3 (b) (ML exact seq) より ok

$$\begin{tikzcd}
0 \arrow[r] & R\lim_s Q^s \arrow[r] \arrow[d, "\cong", "(a)"'] \arrow[dr, phantom, "\circlearrowleft"] & RA^\infty \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \lim_s RQ^s \arrow[r] \arrow[d, "\cong(b)"] & 0 \\
0 \arrow[r] & R\lim_s \overline{Q}^s \arrow[r] & R\overline{A}^\infty \arrow[r] & \lim_s R\overline{Q}^s \arrow[r] & 0
\end{tikzcd}$$

//

§4.4 Whole-plane spectral sequences



平面全体に広がっている場合を扱う.

Recall

§3.6 において. sequence $\{A^s\}_s$ に対して
次の2つを導入した:

- double filtration $\{K_n \mathrm{Im}^r A^s\}_{n,r}$ of A^s
  $$K_n \mathrm{Im}^r A^s := K_n A^s \cap \mathrm{Im}^r A^s$$
- $W := \operatorname{colim}_s R\lim_r K_\infty \mathrm{Im}^r A^s$

whole-plane spectral seq では.
RE_∞ だけでなく W も収束の obstruction となる

Thm 4.4.1 [Boa, Thm 8.2]

Assume
$\{E_r\}$ converges conditionally to $A^{-\infty}$
(resp. A^{∞})
Then
$RE_\infty = 0$ and $W = 0$
$\Rightarrow$ $\{E_r\}$ converges strongly to A^{∞} by $j \circ (\eta^s)^{-1}$
(resp. A^{∞} by $k^{-1} \circ \varepsilon^{s+1}$)

(see Thm 4.4.8 and Thm 4.4.10 for the proof)

Rmk 4.4.2

(Rmk 4.3.2 とほとんど同じ)

- 十分条件については次を参照:
  - $RE_\infty = 0$: Prop 4.1.4, Cor 4.1.8
  - $\boxed{W = 0}$: Prop 4.4.3
- Thm 4.4.1 により、strong convergence は
以下の2つの問題に分割される:
  - ● conditional convergence:
    - structural condition
    - holds for large classes of s.s.
  - ● $RE_\infty = 0$ and $\boxed{W = 0}$:
    - depends only on the data internal to s.s.
    - cannot be expected to hold in general

● Criterion for $W = 0$

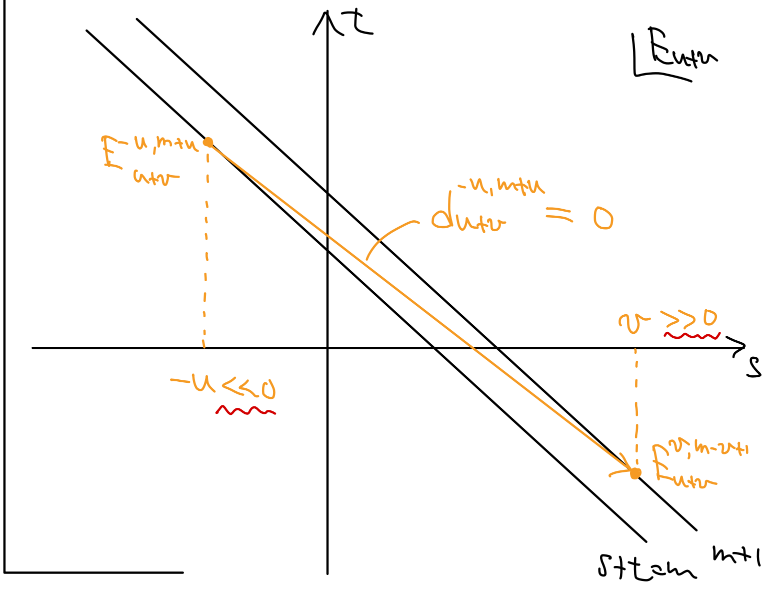
Prop 4.4.3 [Boa, Lem 8.1]

Consider the case
$$\deg i = \deg j = 0, \quad \deg k = +1$$
$$(\rightsquigarrow d_r^{st} : E_r^{st} \longrightarrow E_r^{s+r,\, t-r+1})$$
Assume
$\forall m \in \mathbb{Z}$, ${}^\exists u(m), v(m) \in \mathbb{Z}$ s.t.
$\forall u \geq u(m)$, $\forall v \geq v(m)$,
$$d_{u+v}^{-u,m+u} = 0 : E_{u+v}^{-u,m+u} \longrightarrow E_{u+v}^{v,m-v+1}$$
Then
$W = 0$

Rmk 4.4.4

← deg i, deg j, deg k を指定しない.

- degreewise でなくても成立する
(何故 Boardman はここだけ degreewise にした…?)
- assump は下図のように書ける:



Cor 4.4.5

Assume exiting diff or entering diff
Then $W = 0$

Prop 4.4.3 の証明の準備として.
$\{K_n \mathrm{Im}^r A^s\}$ と spectral seq の関係を調べておく.
(§3.6 では seq $\{A^s\}$ だけであった)

Recall

$$Z_r^s := k^{-1}(\mathrm{Im}^{r-1} A^{s+1})$$
$$B_r^s := j(K_{r-1} A^s)$$

$$E_r^s := Z_r^s / B_r^s$$

Lem 4.4.6 [Boa, (0.7)]

$$\mathrm{Im}\left(d_r^{s-r} : E_r^{s-r} \longrightarrow E_r^s\right) \underset{(1)}{=} B_{r+1}^s / B_r^s \xleftarrow[(2)]{\cong} Z_r^{s-r} / Z_{r+1}^{s-r} \xrightarrow[(3)]{\cong} \frac{K_1 \mathrm{Im}^{r-1} A^{s-r+1}}{K_1 \mathrm{Im}^r A^{s-r+1}}$$

Moreover, the composition of (2) and (3) is described as follows :

$$\begin{array}{ccc} \dfrac{K_1 \mathrm{Im}^{r-1} A^{s-r+1}}{K_1 \mathrm{Im}^r A^{s-r+1}} & \longrightarrow & B_{r+1}^s / B_r^s \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\, i^{r-1}(x) \,] & \longmapsto & [\, j(x) \,] \end{array}$$

(with $i^r(x) = 0$)

proof (1) Prop 2.2.7 (2)

(2) $$Z_r^{s-r} / Z_{r+1}^{s-r} \xrightarrow{\cong} \frac{Z_r^{s-r} / B_r^{s-r}}{Z_{r+1}^{s-r} / B_r^{s-r}} \overset{\text{Prop 2.2.7 (1)}}{=} \frac{E_r^{s-r}}{\mathrm{Ker}\, d_r^{s-r}}$$
$$\xrightarrow[d_r^{s-r}]{\cong} \mathrm{Im}\, d_r^{s-r} \overset{(1)}{=} B_{r+1}^s / B_r^s$$

(3) $k : Z_r^{s-r} \longrightarrow K_1 \mathrm{Im}^{r-1} A^{s-r+1}$: surj

($\because$ $\mathrm{Ker}\, i = \mathrm{Im}\, k$)

$$k^{-1}(K_1 \mathrm{Im}^r A^{s-r+1}) = k^{-1}(\mathrm{Im}^r A^{s-r+1}) = Z_{r+1}^{s-r}$$

compositionはそのまま計算するだけ (see Def 2.2.6) //

double filtration $\{K_n \mathrm{Im}^r A^s\}_{n,r}$ の minimal subquotient は、"internal data" で書ける：

Lem 4.4.7 [Boa, Lem 8.4]

$\forall n \geq 1$, $\forall r \geq 0$

$$\frac{K_n \mathrm{Im}^r A^s}{K_n \mathrm{Im}^{r+1} A^s + K_{n-1} \mathrm{Im}^r A^s} \xrightarrow{\cong} \mathrm{Im}\left(d_{n+r}^{s-n} : E_{n+r}^{s-n} \to E_{n+r}^{s+r}\right)$$
$$[\, i^r(x) \,] \longmapsto [\, j(x) \,]$$

(with $i^{r+n}(x) = 0$)

proof

$i^{n-1} : A^s \longrightarrow A^{s-n+1}$ induces

$$\frac{K_n \mathrm{Im}^r A^s}{K_n \mathrm{Im}^{r+1} A^s + K_{n-1} \mathrm{Im}^r A^s} \xrightarrow{\cong} \frac{K_1 \mathrm{Im}^{r+n-1} A^{s-n+1}}{K_1 \mathrm{Im}^{r+n} A^{s-n+1}}$$

[∵ well-def'd 明らか

surj 明らか

inj $\forall y \in K_n \mathrm{Im}^r A^s$ with $i^{n-1}(y) \in K_1 \mathrm{Im}^{r+n} A^{s-n+1}$

(i.e. $y \in \mathrm{Im}^r A^s$, $i^n(y) = 0$,
$\exists z \in A^{s+r+1}$ s.t. $i^{n-1}(y) = i^{r+n}(z)$)

とする。このとき

$$i^{n-1}(y - i^{r+1}(z)) = 0$$

$\leadsto$ $y - i^{r+1}(z) \in K_{n-1} \mathrm{Im}^r A^s$
($\because$ $y \in \mathrm{Im}^r A^s$)

$\leadsto$ $y \in K_n \mathrm{Im}^{r+1} A^s + K_{n-1} \mathrm{Im}^r A^s$

($\because$ $i^n(i^{r+1}(z)) = i(i^{r+n}(z))$
$= i^n(y) = 0$)

よって inj.]

よって Lem 4.4.6 の isom と合成することで、欲しい isom を得る

((r in Lem 4.4.6) := r + n
(s ——〃——) := s + r) //

proof of Prop 4.4.3

Fix $m \in \mathbb{Z}$ ($\leadsto$ prove $W^m = 0$)

Define

$$\begin{cases} r_0 := u(m) + v(m) - 1 \\ s := -u(m) + 1 \end{cases}$$

By Lem 3.6.9, enough to show :

[$1 \leq \forall n < \infty$, $\forall r \geq r_0$,

$$\frac{K_n \mathrm{Im}^r A^{s, m-s}}{K_n \mathrm{Im}^{r+1} A^{s, m-s} + K_{n-1} \mathrm{Im}^r A^{s, m-s}} = 0$$

Fix $1 \leq n < \infty$, $r \geq r_0$.

Define

$$\begin{cases} u := n + u(m) - 1 & (\geq u(m)) \\ v := r - u(m) + 1 & (\geq v(m)) \end{cases}$$

By Lem 4.4.7,

$$\frac{K_n \mathrm{Im}^r A^{s, m-s}}{K_n \mathrm{Im}^{r+1} + K_{n-1} \mathrm{Im}^r} = \mathrm{Im}\left(d_{\underset{u+v}{\underset{\shortparallel}{\boxed{n+r}}}}^{\overset{-u}{\overset{\shortparallel}{\boxed{s-n}}}, m-s+n}\right) \overset{\text{assump}}{=} 0$$

//

Thm 4.4.1 は、colim と lim 別々に証明する

◎ colim as target

Thm 4.4.8 [Boa, Thm 8.10]

2 of the following ⇒ 3rd :

(1) $\{E_r\}$ converges conditionally to $A^{-\infty}$
 (i.e. $A^\infty = RA^\infty = 0$)

(2) $RE_\infty = 0$ and $W = 0$

(3) $\{E_r\}$ converges strongly to A^∞ by $j^0 (h^0)^{-1}$

entering diff のとき (Thm 4.3.5) は、isom

$$\begin{cases} Q^0 \xrightarrow{\cong} F^\infty A^{-\infty} \\ RQ^0 \xrightarrow{\cong} RF^\infty A^{-\infty} \end{cases} \quad (\text{Lem } 4.3.4)$$

が重要であった。

これの一般化 (+α) を与える。

Lem 4.4.9

Assume

- $RE_\infty = 0$ and $W = 0$
- $\{E_r\}$ converges weakly to A^∞

Then

- $A^\infty \xrightarrow{\cong} \operatorname{colim}_s Q^s \xrightarrow{\cong} F^\infty A^{-\infty}$
- $RA^\infty \xrightarrow{\cong} \operatorname{colim}_s RQ^s \xrightarrow{\cong} RF^\infty A^{-\infty}$

proof

By Lem 3.6.7 ← $W=0$

$$\begin{cases} \operatorname{colim}_s Q^s \xrightarrow{\cong} F^\infty A^\infty \\ \operatorname{colim}_s RQ^s \xrightarrow{\cong} RF^\infty A^\infty \end{cases}$$

By Lem 4.1.23 ← $RE_\infty = 0$

(b) $\forall s,\ RA^\infty \xrightarrow{\cong} RQ^s$

$\rightsquigarrow RA^\infty \xrightarrow{\cong} \operatorname{colim}_s RQ^s$

(c) ((1) ⇒ (3)) ← weakly conv.

$\forall s,\ A^\infty \xrightarrow{\cong} Q^s$

$\rightsquigarrow A^\infty \xrightarrow{\cong} \operatorname{colim}_s Q^s$

//

Lem 4.3.4 の代わりに Lem 4.4.9 を使えば、Thm 4.3.5 と同じ方針で証明できる:

proof of Thm 4.4.8

Note that

$\{F^s A^{-\infty}\}_s$: exhaustive (∵ Lem 4.1.17(a))

(1)(2) ⇒ (3)

By Lem 4.1.23 (c) ((2) ⇒ (1)), ← (1) $A^\infty = 0$ (2) $RE_\infty = 0$

$\{E_r\}$: weakly convergent to A^∞ — ①

By Lem 4.4.9, ← (2) ①

$$\begin{cases} F^\infty A^{-\infty} \cong A^\infty = 0 \\ RF^\infty A^{-\infty} \cong RA^\infty = 0 \end{cases}$$

← (1)

i.e. $\{F^s A^{-\infty}\}_s$: complete Hausdorff

(2)(3) ⇒ (1)

By Lem 4.4.9, ← (2) (3) weak conv.

$$\begin{cases} A^\infty \cong F^\infty A^{-\infty} = 0 \\ RA^\infty \cong RF^\infty A^{-\infty} = 0 \end{cases}$$

← (3) Hausdorff

← (3) complete

(3)(1) ⇒ (2)

By Lem 4.1.26 ← (1) $RA^\infty = 0$

- $\forall s,\ RQ^s = 0$ — ②
- $0 \to \dfrac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \xrightarrow{\cong} E_\infty^s \to Q^{s+1} \to Q^s \to RE_\infty^s \to 0$ — ③ exact

← (3) weak conv.

$\rightsquigarrow \forall s,\ Q^{s+1} \to Q^s$: inj — ④

By Lem 3.6.7

$0 \to \operatorname{colim}_s Q^s \to \overset{0}{\cancel{F^\infty}} A^{-\infty} \to W$ ← (3) Hausdorff

$\to \operatorname{colim}_s RQ^s \to \overset{0}{\cancel{RF^\infty}} A^{-\infty} \to 0$: exact ← (3) complete

$$\rightsquigarrow \begin{cases} \cdot\ \operatorname{colim}_s Q^s = 0 \quad \text{— ⑤} \\ \cdot\ W \cong \operatorname{colim}_s RQ^s \underset{②}{=} 0 \end{cases}$$

By ④⑤,

$\forall s,\ Q^s = 0$

$\underset{③}{\rightsquigarrow} \forall s,\ RE_\infty^s = 0$

//

(ii) lim as target

Thm 4.4.10 [Boa, Thm 8.13]

Assume

$\{E_r\}$ converges conditionally to A^∞

(i.e. $A^{-\infty}=0$)

Then TFAE:

(1) $RE_\infty=0$ and $W=0$

(2) $RA^\infty=0$ and converges strongly to A^∞ by $k^{-1}\circ\varepsilon^{s+1}$

proof

By Lem 4.1.17, $A^{-\infty}=0$

$\{F^sA^\infty\}$: complete Hausdorff, exhaustive

By Lem 4.1.19 (3), ← $A^{-\infty}=0$

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^s \arrow[r] & Q^{s+1} \arrow[r] & Q^s \arrow[r] & RE_\infty^s \arrow[r] & RQ^{s+1} \arrow[r] & RQ^s \arrow[r] & 0 \\
0 \arrow[r] & \frac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[u, "k^{-1}\circ\varepsilon^{s+1}"] \arrow[r] & \mathrm{Im}^\infty A^{s+1} \arrow[u, hook] \arrow[r] & \mathrm{Im}^\infty A^s \arrow[u, hook] \arrow[r] & 0
\end{tikzcd}$$

exact

: exact — ①

(1) ⇒ (2)

← (1) $RE_\infty=0$

By Lem 4.1.23 (a) (b),

- $\forall s$, $RA^\infty \xrightarrow{\cong} RQ^s$

  ⇝ $RA^\infty \overset{\text{Lem 3.6.8, } A^{-\infty}=0}{\cong} \operatorname{colim}_s RQ^s = W \overset{(1)}{=} 0$
- $\forall s$, $\mathrm{Im}^\infty A^s = Q^s$ ——— ②

①② + 5-lemma ⇝ $k^{-1}\circ\varepsilon^{s+1}$: isom

⇝ strong conv.

(2) ⇒ (1)

$W=0$

(2) $RA^\infty=0$

By Cor 3.5.5, $\forall s$, $RQ^s=0$ — ③

⇝ $W = \operatorname{colim}_s RQ^s = 0$

↑ Lem 3.6.8, $A^{-\infty}=0$

$E_\infty=0$

$C^s := Q^s/\mathrm{Im}^\infty A^s$

⇝ $\forall s$, $C^s=0$ を示したい

(entering diff のとき (Thm 4.3.6) は induction on s で示せたが、今回は colim_s を扱う必要がある)

$\forall s$, $C^{s+1}\to C^s$: inj ——— ④

(∵ Apply Lem 3.2.15 to ①

$$\begin{tikzcd}
0 \arrow[r] & C^{s+1} \arrow[r] & C^s \\
E_\infty^s \arrow[u] \arrow[r] & Q^{s+1} \arrow[u] \arrow[r] & Q^s \arrow[u] \arrow[r] & RE_\infty^s \arrow[r] & RQ^{s+1} \\
\frac{F^sA^\infty}{F^{s+1}A^\infty} \arrow[u, "\cong"] \arrow[r] & \mathrm{Im}^\infty A^{s+1} \arrow[u, hook] \arrow[r] & \mathrm{Im}^\infty A^s \arrow[u, hook] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \\
 & & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

(2) weak conv.)

$\operatorname{colim}_s C^s=0$ ——— ⑤

(∵ • $Q^s \hookrightarrow A^s$: inj

$\xrightarrow{\operatorname{colim}_s}$ $\operatorname{colim}_s Q^s \to \operatorname{colim}_s A^s$: inj

$\operatorname{colim}_s Q^s = 0$ ← exact

$\operatorname{colim}_s A^s = A^{-\infty} = 0$ ← assump

• $Q^s \twoheadrightarrow C^s$: surj

$\xrightarrow{\operatorname{colim}_s}$ $\operatorname{colim}_s Q^s \to \operatorname{colim}_s C^s$: surj

$\operatorname{colim}_s Q^s = 0$, $\operatorname{colim}_s C^s = 0$)

By ④⑤,

$\forall s$, $C^s=0$

i.e. $\forall s$, $\mathrm{Im}^\infty A^s = Q^s$

$\overset{①}{\leadsto}$ $\forall s$, $Q^{s+1}\to Q^s$: surj — ⑥

By ①③⑥,

$\forall s$, $RE_\infty^s=0$

//

Rmk 4.4.11

(1) ⇒ (2) において、

strong conv を示すだけなら $W=0$ は不要.

しかし、実用上は $RA^\infty=0$ もあった方が良い

(cf. Thm 3.3.2 (2)

$0\to R\lim_s H^{n-1}(C^s)\to H^n(\lim_s C^s)\to \lim_s H^n(C^s)\to 0$

exact)

◎ Comparison theorem

Thm 4.4.12 [Boa, Thm 8.3]

$f : (A^s, E^s) \to (\bar{A}^s, \bar{E}^s)$: morph of unrolled exact couple

Assume

- $\{E_r\}$ and $\{\bar{E}_r\}$ conditionally converges to either (1) or (2) :
  - (1) $G = A^{-\infty}$ and $\bar{G} = \bar{A}^{-\infty}$ ← $R\bar{A}^{\infty} = 0$ は不要
  - (2) $G = A^{\infty}$ and $\bar{G} = \bar{A}^{\infty}$
- $f_\infty : E_\infty \xrightarrow{\cong} \bar{E}_\infty$: isom

  $Rf_\infty : RE_\infty \xrightarrow{\cong} R\bar{E}_\infty$: isom

  $f_W : W \xrightarrow{\cong} \bar{W}$: isom

Then

(a) $\forall s,\ Q^s \xrightarrow{\cong} \bar{Q}^s$: isom

(b) $\forall s,\ RQ^s \xrightarrow{\cong} R\bar{Q}^s$: isom

(c) $G \xrightarrow{\cong} \bar{G}$: isom of filtered modules

(i.e. · $G \xrightarrow{\cong} \bar{G}$ as modules
· $\forall s,\ F^sG \xrightarrow{\cong} F^s\bar{G}$)

(d) $RA^{\infty} \xrightarrow{\cong} R\bar{A}^{\infty}$

proof in the case (1)

($R\bar{A}^{\infty} = 0$ は仮定しない)

$RA^{\infty} = 0$

Lem 4.1.26 (for (A^s, E^s)), Lem 4.1.19 (for $(\bar{A}^s, \bar{E}^s)$) より.

$$\begin{tikzcd}
0 \arrow[r] & \frac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E^s_\infty \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & Q^{s+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Q^s \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & RE^s_\infty \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[d] & & \text{exact} \\
0 \arrow[r] & \frac{F^s\bar{A}^{-\infty}}{F^{s+1}\bar{A}^{-\infty}} \arrow[r] & \bar{E}^s_\infty \arrow[r] & \bar{Q}^{s+1} \arrow[r] & \bar{Q}^s \arrow[r] & R\bar{E}^s_\infty \arrow[r] & R\bar{Q}^{s+1} \arrow[r] & R\bar{Q}^s \arrow[r] & 0
\end{tikzcd}$$

—— ①

exact

(a) $Q^s \xrightarrow{\cong} \bar{Q}^s$

Thm 4.3.7 と全く同じ証明で ok.

(entering diff という仮定を使っていなかった)

(c) $A^{-\infty} \xrightarrow{\cong} \bar{A}^{-\infty}$

① と (a) より,

$$\forall s,\quad \frac{F^sA^{-\infty}}{F^{s+1}A^{-\infty}} \xrightarrow{\cong} \frac{F^s\bar{A}^{-\infty}}{F^{s+1}\bar{A}^{-\infty}}$$

よって Thm 3.4.6 より,

$A^{-\infty} \xrightarrow{\cong} \bar{A}^{-\infty}$: isom of filtered modules

∵ Thm 3.4.6 の仮定を check する

$\{F^sA^{-\infty}\}, \{F^s\bar{A}^{-\infty}\}$: exhaustive

Lem 4.1.17 (a) より常に成立

$F^{\infty}A^{-\infty} \xrightarrow{\cong} F^{\infty}\bar{A}^{-\infty}$ ((c) の変更点はここだけ)

Lem 4.1.26 より. ← $RA^{\infty} = 0$

$\forall s,\ RQ^s = 0$

よって Lem 3.6.7 より,

assump

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"'] & \operatorname{colim}_s Q^s \arrow[r] \arrow[d, "(a)\,\cong"'] & F^{\infty}A^{-\infty} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[d] & \text{exact} \\
0 \arrow[r] & \operatorname{colim}_s \bar{Q}^s \arrow[r] & F^{\infty}\bar{A}^{-\infty} \arrow[r] & \bar{W} \arrow[r] & \operatorname{colim}_s R\bar{Q}^s
\end{tikzcd}$$

exact

$\xrightarrow{\text{5-lemma}}$ $F^{\infty}A^{-\infty} \xrightarrow{\cong} F^{\infty}\bar{A}^{-\infty}$: isom

$\{F^sA^{-\infty}\}$: complete

$RA^{\infty} = 0$ なので Lem 4.1.26 (b) より ok

← assump

(b) $RQ^s \xrightarrow{\cong} R\bar{Q}^s$

Cor 3.5.5 より, ← $RA^{\infty} = 0$

$\forall s,\ RQ^s = 0$

(→ 「$\forall s,\ R\bar{Q}^s = 0$」を示せば ok)

ここで

$\operatorname{colim}_s R\bar{Q}^s = 0$ —— ②

(b) の変更点

∵ 再び Lem 3.6.7 より,

$$\begin{tikzcd}
F^{\infty}A^{-\infty} \arrow[r] \arrow[d, "(c)\,\cong"'] & W \arrow[r] \arrow[d, "\cong"'] & 0 \\
F^{\infty}\bar{A}^{-\infty} \arrow[r] & \bar{W} \arrow[r] & \operatorname{colim}_s R\bar{Q}^s \arrow[r] & RF^{\infty}\bar{A}^{-\infty} \arrow[r] & 0
\end{tikzcd}$$

exact

$$\begin{tikzcd}
\arrow[r] & \operatorname{colim}_s R\bar{Q}^s \arrow[r, "\cong"] & RF^{\infty}\bar{A}^{-\infty} & \text{: isom} \\
& & RF^{\infty}A^{-\infty} = 0 \arrow[u, "(c)\,\cong"']
\end{tikzcd}$$

↑ Lem 4.1.26 (b), $RA^{\infty} = 0$

また, ① より

$\forall s,\ R\bar{Q}^{s+1} \xrightarrow{\cong} R\bar{Q}^s$ —— ③

②③ より,

$\forall s,\ R\bar{Q}^s = 0$

(d) $RA^{\infty} \xrightarrow{\cong} R\bar{A}^{\infty}$

(a) (b) と Thm 3.5.3 (b) (ML exact seq) より ok

(Thm 4.3.7 と全く同じ)

//

proof in the case (2)

Lem 4.1.19 (1) より

$$\begin{tikzcd}
A^{-\infty}=0 \arrow[d] & 0 \arrow[r]\arrow[d] & K^{s+1} \arrow[r]\arrow[d] & K^{s} \arrow[r]\arrow[d] & 0 \arrow[r]\arrow[d] & L^{s+1} \arrow[r]\arrow[d] & L^{s} \arrow[r]\arrow[d] & 0 \arrow[d] & \leftarrow \mathrm{Ker} \\
0 \arrow[r] & E^{s}_{\infty} \arrow[r]\arrow[d, "\cong"']\arrow[dr, phantom, "\circlearrowleft"] & Q^{s+1} \arrow[r]\arrow[d]\arrow[dr, phantom, "\circlearrowleft"] & Q^{s} \arrow[r]\arrow[d]\arrow[dr, phantom, "\circlearrowleft"] & RE^{s}_{\infty} \arrow[r]\arrow[d, "\cong"']\arrow[dr, phantom, "\circlearrowleft"] & RQ^{s+1} \arrow[r]\arrow[d]\arrow[dr, phantom, "\circlearrowleft"] & RQ^{s} \arrow[r]\arrow[d] & 0 \arrow[d] & :\mathrm{exact} \arrow[d, phantom, "\text{— ④}"] \\
0 \arrow[r] & \bar{E}^{s}_{\infty} \arrow[r]\arrow[d] & \bar{Q}^{s+1} \arrow[r]\arrow[d] & \bar{Q}^{s} \arrow[r]\arrow[d] & R\bar{E}^{s}_{\infty} \arrow[r]\arrow[d] & R\bar{Q}^{s+1} \arrow[r]\arrow[d] & R\bar{Q}^{s} \arrow[r]\arrow[d] & 0 \arrow[d] & :\mathrm{exact} \\
\bar{A}^{-\infty}=0 \arrow[u] & 0 \arrow[r] & C^{s+1} \arrow[r] & C^{s} \arrow[r] & 0 \arrow[r] & D^{s+1} \arrow[r] & D^{s} \arrow[r] & 0 & \leftarrow \mathrm{Coker}
\end{tikzcd}$$

(a) $Q^s \xrightarrow{\cong} \bar{Q}^s$

$K^s := \mathrm{Ker}(Q^s \longrightarrow \bar{Q}^s)$, $C^s := \mathrm{Coker}(Q^s \rightarrow \bar{Q}^s)$ とおく.

④ と Lem 3.2.15 より,

$$\begin{cases} \forall s,\ K^{s+1} \xrightarrow{\cong} K^s : \mathrm{isom} \\ \forall s,\ C^{s+1} \hookrightarrow C^s : \mathrm{inj} \end{cases} \quad \text{— ⑤}$$

ここで

$$\operatorname*{colim}_s K^s = \operatorname*{colim}_s C^s = 0$$

(a)の変更点

cond. conv.

∵ $Q^s \hookrightarrow A^s$, $\bar{Q}^s \hookrightarrow \bar{A}^s$: inj と $A^{-\infty} = \bar{A}^{-\infty} = 0$ より,

$$Q^{-\infty} = \bar{Q}^{-\infty} = 0$$

$$0 \rightarrow K^s \rightarrow Q^s \longrightarrow \bar{Q}^s \rightarrow C^s \rightarrow 0 : \mathrm{exact}$$

$\left\{ \operatorname*{colim}_s : \mathrm{exact} \right.$

$$0 \rightarrow K^{-\infty} \rightarrow \underset{\overset{\shortparallel}{0}}{Q^{-\infty}} \rightarrow \underset{\overset{\shortparallel}{0}}{\bar{Q}^{-\infty}} \rightarrow C^{-\infty} \rightarrow 0 : \mathrm{exact}$$

よって ⑤ より,

$$\forall s,\ K^s = C^s = 0$$

(b) $RQ^s \xrightarrow{\cong} R\bar{Q}^s$

$L^s := \mathrm{Ker}(RQ^s \rightarrow R\bar{Q}^s)$, $D^s := \mathrm{Coker}(RQ^s \rightarrow R\bar{Q}^s)$

(a), ④, Lem 3.2.15 より,

$$\begin{cases} \forall s,\ L^{s+1} \xrightarrow{\cong} L^s : \mathrm{isom} \\ \forall s,\ D^{s+1} \xrightarrow{\cong} D^s : \mathrm{isom} \end{cases} \quad \text{— ⑥}$$

ここで

$$\operatorname*{colim}_s L^s = \operatorname*{colim}_s D^s = 0$$

(b)の変更点

∵ $\operatorname*{colim}_s$: exact より

$$\operatorname*{colim}_s L^s = \operatorname*{colim}_s \left(\mathrm{Ker}(RQ^s \rightarrow R\bar{Q}^s) \right)$$

$$= \mathrm{Ker}\left(\operatorname*{colim}_s RQ^s \longrightarrow \operatorname*{colim}_s R\bar{Q}^s\right)$$

Lem3.6.8

$A^{\infty} = \bar{A}^{-\infty} = 0$

$$= \mathrm{Ker}(W \xrightarrow[\cong]{} \bar{W})$$

← assump.

$$= 0$$

$\operatorname*{colim}_s D^s$ についても同様

(Ker を Coker に置きかえるだけ)

よって ⑥ より,

$$\forall s,\ L^s = D^s = 0$$

(c) $A^{\infty} \longrightarrow \bar{A}^{\infty}$

(d) $RA^{\infty} \longrightarrow R\bar{A}^{\infty}$

} Thm 4.3.7 と全く同じ証明で ok

//

Rmk 4.4.13 (疑問点)

$\exists r_0$, $f_{r_0} : E_{r_0} \xrightarrow{\cong} \bar{E}_{r_0}$: isom

と仮定する.

このとき Prop 4.1.9 より

$$\begin{cases} f_\infty : E_\infty \xrightarrow{\cong} \bar{E}_\infty \\ Rf_\infty : RE_\infty \xrightarrow{\cong} R\bar{E}_\infty \end{cases} : \mathrm{isom}$$

であった.

f_w についても成立するのだろうか?

([Boa] には書いてなかったし,
あまり必要性を感じないので保留とする)

② Non-convergent example

Example 4.4.14 [Boa, Example in p.26]

(1) $\{A^s\}_s$: as in Example 3.6.10 concentrated in deg 0

$$\left(\begin{array}{l} \text{For } n \geq 0, \\ A^{n,-n} = A^{-n,n} = \bigoplus_{t \geq n} \mathbb{K} x_t \end{array} \right)$$

Define

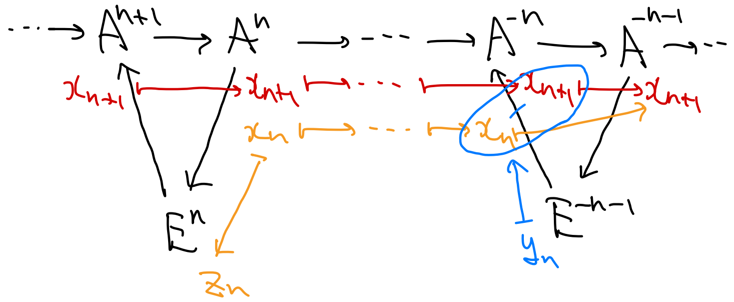
$$E^{st} := \begin{cases} \operatorname{Coker}(A^{s+1,-(s+1)} \to A^{s,-s}) & (s+t=0) \\ \operatorname{Ker}(\quad \text{—}\,\text{''}\,\text{—} \quad) & (s+t=-1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$= \begin{cases} \mathbb{K} z_n & ((s,t)=(n,-n)) \\ \mathbb{K} y_n & ((s,t)=(-n-1,n)) \\ 0 & (\text{otherwise}) \end{cases} \quad (n \geq 0)$$

$$\left(\begin{array}{l} \underline{\text{where}} \\ z_n := [x_n] \in A^n / i(A^{n+1}) \\ y_n := x_n - x_{n+1} \in A^{-n} \end{array} \right)$$

Then we have

(A^s, E^s) : unrolled exact couple with $\deg i = \deg j = 0$, $\deg k = 1$



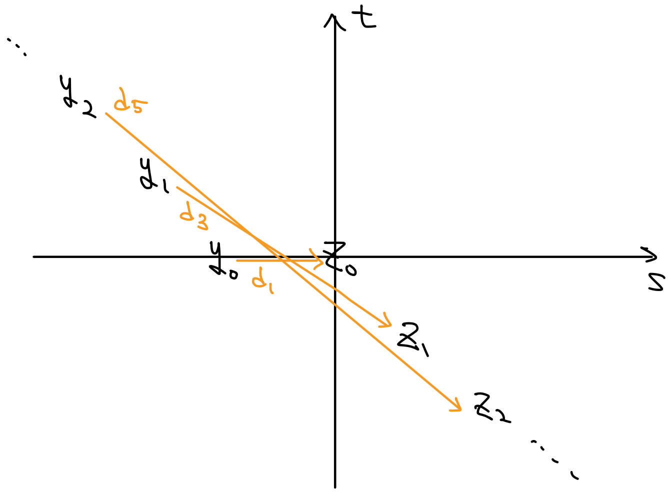
By def of E_r and d_r, we have

$$\begin{array}{ccc} d_{2n+1}^{-n-1,n} : E_{2n+1}^{-n-1,n} & \longrightarrow & E_{2n+1}^{n,-n} \\ \text{\rotatebox{90}{=}} & & \text{\rotatebox{90}{=}} \\ \mathbb{K} y_n & \longrightarrow & \mathbb{K} z_n \\ y_n & \longmapsto & z_n \end{array}$$

⇝ everything will be killed

Hence

$$E_\infty = RE_\infty = 0$$



⇝ hypothesis of Prop 4.4.3 fails

But this is NOT conditionally conv.

(∵ $RA^\infty \neq 0$)

(2) To obtain a conditionally conv. example, we complete $\{A^s\}$ as in Def 3.5.9

$$\hat{A}^s := \lim_r A^s / \operatorname{Im}^r A^s$$

Then we have :

- $\hat{A}^n = \hat{A}^{-n} = \prod_{t \geq n} \mathbb{K} x_t \quad (n \geq 0)$
- $\hat{A}^\infty = R\hat{A}^\infty = 0$ (∵ Lem 3.5.10 (2))

  ⇝ conditionally conv. to $\hat{A}^{-\infty}$
- $\hat{A}^{-\infty} = \hat{A}^0 / K \neq 0$

  $$\left(\begin{array}{l} \underline{\text{where}} \\ K := \operatorname{Ker}\left(\bigoplus_{t \geq 0} \mathbb{K} x_t \xrightarrow{\text{sum}} \mathbb{K}\right) \\ \quad \subset \bigoplus_{t \geq 0} \mathbb{K} x_t \subset \prod_{t \geq 0} \mathbb{K} x_t = \hat{A}^0 \end{array} \right)$$
- $\hat{E}_r^s = E_r^s$

  ⇝ $\hat{E}_\infty^s = R\hat{E}_\infty^s = 0$

Thus

$\{\hat{E}_r\}$ is NOT strongly conv. to $\hat{A}^{-\infty}$

⇝ Thm 4.4.8 fails without the hypothesis $W = 0$

Rmk 4.4.14

<u>上のような議論</u>がうまくいくのは, $\forall s$, $\begin{cases} Q^{s+1} \xrightarrow{\cong} Q^s \\ RQ^{s+1} \xrightarrow{\cong} RQ^s \end{cases}$ のときに限るらしい

$$\left(\begin{array}{l} \{A^s\} \rightsquigarrow \{\hat{A}^s\} : \text{completion} \\ \quad \rightsquigarrow \hat{E}^s := \operatorname{Ker} \oplus \operatorname{Coker} \end{array} \right)$$

§5 Examples

(§1.2とは degree がズレていることに注意)

§5.1 Revisit examples in §1.2 [NoRef]

$$\cdots \longrightarrow M^s \xrightarrow{f^s} M^{s-1} \xrightarrow{f^{s-1}} \cdots \longrightarrow M^1 \xrightarrow{f^1} M^0$$

seq of (ungraded) $\mathbb{K}$-modules

(Define $M^{-1} = 0$, $f^0 = 0 : M^0 \to 0$)

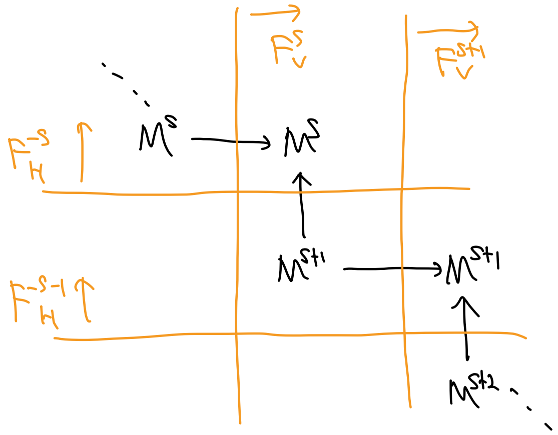
Def 5.1.1

- Define a double cpx $\{D^{st}\}$ by

$$\begin{tikzcd}
\underset{\substack{\shortparallel \\ D^{-1,0}}}{M^0} \arrow[r, "id"] & M^0 = D^{00} & & \\
& \underset{\substack{\shortparallel \\ D^{0,-1}}}{M^1} \arrow[u, "-f^1"] \arrow[r, "id"] & M^1 & \\
& & M^2 \arrow[u, "-f^2"] \arrow[r, "id"] & M^2 \arrow[dr, phantom, "\ddots"] \\
& & & &
\end{tikzcd}$$

- $\mathrm{Tot}^n D := \bigoplus_{i+j=n} D^{ij}$

$$\begin{cases} F^s_V = F^s_V \mathrm{Tot}\, D := \bigoplus_{i \geq s} D^{ij} \\ F^t_H = F^t_H \mathrm{Tot}\, D := \bigoplus_{j \geq t} D^{ij} \end{cases}$$ subcpx



$$\begin{cases} \cdot\ \cdots \subset F^{s+1}_V \subset F^s_V \subset \cdots \subset F^0_V \subset F^{-1}_V = \mathrm{Tot}D \\ \cdot\ 0 = F^1_H \subset F^0_H \subset F^{-1}_H \subset \cdots \subset F^{-t}_H \subset F^{-t-1}_H \subset \cdots \subset \mathrm{Tot}D \end{cases}$$

まずは cohom を計算しておく.

Lem 5.1.2

(1) $\forall t$, $H^*(F^t_H) = 0$

(2) $H^*(\mathrm{Tot}\, D) = 0$

(3) $\forall s \geq 0$, $H^k(F^s_V) \cong \begin{cases} M^s & (k=0) \\ 0 & (k \neq 0) \end{cases}$

Moreover, $M^s = D^{s,-s} \hookrightarrow F^s_V$ induces $M^s \xrightarrow{\cong} H^0(F^s_V)$

(4)
$$\begin{tikzcd}
H^0(F^s_V) \arrow[r] & H^0(F^{s-1}_V) \\
M^s \arrow[u, "\cong"] \arrow[r, "f^s"] \arrow[ur, phantom, "\circlearrowleft"] & M^{s-1} \arrow[u, "\cong"']
\end{tikzcd}$$

proof (いずれも直接計算でもできるけど、少し工夫する)

(1) induction on $t \leq 1$

$\underline{t=1}$ $F^1_H = 0$ なので明らか.

$\underline{t \leq 0}$ $F^t_H / F^{t+1}_H \cong (M^t \xrightarrow{\cong} M^t)$ as cpx

$\rightsquigarrow H^*(F^t_H / F^{t+1}_H) = 0$

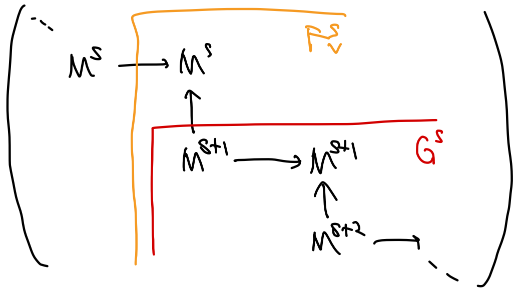
これと $H^*(F^{t+1}_H) = 0$ (ind. hyp.) より ok.

$\left(0 \to F^{t+1}_H \to F^t_H \to F^t_H / F^{t+1}_H \to 0 : \text{exact} \right)$

(2) $\mathrm{Tot}D = \operatorname{colim}_t F^t_V$ より、

$$H^*(\mathrm{Tot}D) = H^*(\operatorname{colim}_t F^t_V) = \operatorname{colim}_t H^*(F^t_V) \overset{(1)}{=} 0$$

(3) $G^s := \bigoplus_{j<-s} D^{ij} = F^s_V / M^s$ と def



ここで (2) (と同様の議論) より、

$H^*(G^s) = 0$

よって、$0 \to M^s \to F^s_V \to G^s \to 0$: exact より、

$$\underset{\substack{\shortparallel \\ M^s \text{ at deg } 0}}{H^*(M^s)} \xrightarrow{\cong} H^*(F^s_V) : \text{isom}$$

(4) (3)から直ちに従う.

//

◎ Spectral seq for $\{F^t_H\}_t$

（こっちは自明に収束する）

Define

$$\begin{tikzcd}
\cdots \arrow[r] & A^{t+1}_H \arrow[rr] & & A^t_H \arrow[rr] \arrow[dl] & & A^{t-1}_H \arrow[r] \arrow[dl] & \cdots \\
& & E^t_H \arrow[ul] & & E^{t-1}_H \arrow[ul] & &
\end{tikzcd}$$

: unrolled exact couple

by

$$\begin{cases} \cdot\ A^t_H = \{A^{n-t,t}_H\}_n, & A^{st}_H := H^{s+t}(F^t_H) \\ \cdot\ E^t_H = \{E^{n-t,t}_H\}_n, & E^{st}_H := H^{s+t}(F^t_H / F^{t+1}_H) \end{cases}$$

← bidegree の役割が通常と逆なことに注意.

Then we have

$\{E^{st}_{Hr}, d^{st}_{Hr}\}_{s,t,r}$: spectral sequence

s.t. $d^{st}_{Hr} : E^{st}_{Hr} \longrightarrow E^{s+r+1,t+r}_{Hr}$

（ここまでは double cpx の s.s. の一般論）

Lem 5.1.3

$\forall t,\ A^t_H = E^t_H = 0$

proof Lem 5.1.2 (1) //

もっと一般の状況で言える (Rmk 5.3.6 (1))

Prop 5.1.4

$\overset{=0}{\{E_{Hr}\}_r}$ converges strongly to $\overset{=0}{H^*(\mathrm{Tot} D)}$

proof

Lem 5.1.3 (1) より、

$\{E_{Hr}\}_r$: half-plane s.s. with exiting diff's.

with $A^\infty_H = 0$

よって Thm 4.2.1 より、

$\{E_{Hr}\}_r$ converges strongly to $A^{-\infty} = H^*(\mathrm{Tot} D)$

//

Rmk 5.1.5

実際、

$$\begin{cases} \cdot\ E_{H1} = 0 \rightsquigarrow E_{H\infty} = 0 \\ \cdot\ H^*(\mathrm{Tot} D) = 0 \end{cases}$$

なので、strongly conv. が直接確認できる

($\rightsquigarrow$ Prop 5.1.4 は自明なことをしているだけ!)

◎ Spectral seq. for $\{F^s_V\}_s$

（こっちは一般には収束しない）

Simply we write

$F^s := F^s_V$

← (cf. Ex 5.3.5 completion)

（$\{F^t_H\}_t$ はつまらなかったので今後出てこない）

Define

$$\begin{tikzcd}
\cdots \arrow[r] & A^{s+1} \arrow[rr] & & A^s \arrow[r] \arrow[dl] & \cdots \\
& & E^s \arrow[ul] & &
\end{tikzcd}$$

: unrolled exact couple

by

$$\begin{cases} A^s := H^*(F^s) \\ E^s := H^*(F^s / F^{s+1}) \end{cases}$$

Then we have

$\{E^{st}_r, d^{st}_r\}_{s,t,r}$: spectral sequence

s.t. $d^{st}_r : E^{st}_r \longrightarrow E^{s+r, t-r+1}_r$ ← いつものやつ

Lem 5.1.6

(1) $\forall s < -1,\ E^s = 0$

(2) $\forall s < 0,\ A^s = H^*(\mathrm{Tot} D) = 0$

($\rightsquigarrow A^{-\infty} = 0$)

(3) $A^\infty \cong \lim_s M^s$

$RA^\infty \cong R\lim_s M^s$

proof Lem 5.1.2 //

Ex 5.1.7

$\{M^s\}_s$: as in Ex 1.2.2

(i.e. $\cdots \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \xrightarrow{id} \mathbb{K}$)

Ex 1.2.2 では.（素朴に）$A^{-\infty} = H^*(\mathrm{Tot} D)$ を target にしていた.

Ex 1.2.2 で見たように、

$E_1 \cong \mathbb{K}$, $H^*(\mathrm{Tot} D) = 0$

なので、これは収束していない。

実際、

$A^\infty = \lim M^s = \mathbb{K} \neq 0$

なので、conditionally convergent ですらない

このS.S.を詳しく調べるために.

E_∞, RE_∞ をちゃんと計算する.

Observation 5.1.8

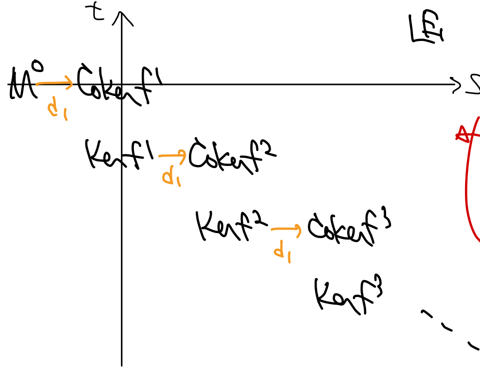
E_1, E_2 あたりで~~様子~~を見る.

- $E_1^{st} = H^{s+t}(F^s/F^{s+1})$

  d_1 : connecting hom for

  $$0 \to F^{s+1}/F^{s+2} \to F^s/F^{s+2} \to F^s/F^{s+1} \to 0 : \text{exact}$$

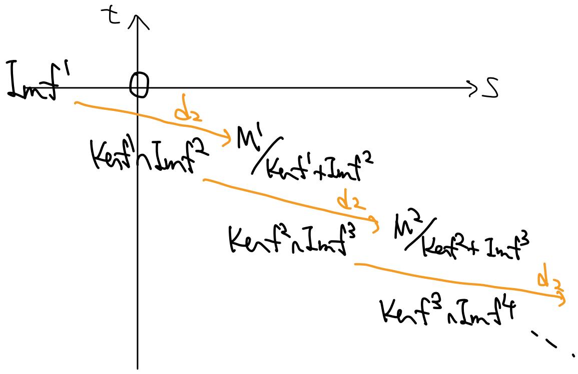
  (see Thm 1.1.10, Cor 2.3.3)



＊(Ex 5.1.7では, $E_1^{-1,0} = \mathbb{K}$ 以外は全て0.)

$$\left(d_1 : \mathrm{Ker} f^s \hookrightarrow M^s \twoheadrightarrow \mathrm{Coker} f^{s+1}\right)$$

- $E_2 = H(E_1)$



d_2 がよく分からないし,

d_3 以降も非自明(かもしれない)

↖ この方針では厳しい.

Recall

$$\begin{cases} Z_r^s = k^{-1}(\mathrm{Im}^{r-1} A^{s+1}) \subset E^s \\ B_r^s = j(K_{r-1} A^s) \subset E^s \end{cases}$$

$$\begin{cases} E_\infty^s = \bigcap_r Z_r^s / \bigcup_r B_r^s \\ RE_\infty^s = R\lim_r Z_r^s \end{cases}$$

↖ これを計算する.

Lem 5.1.9

(1)

$$F^{st} = E_1^{st} = \begin{cases} \mathrm{Coker} f^{s+1} & (s+t=0) \\ \mathrm{Ker} f^{s+1} & (s+t=-1) \\ 0 & (\text{otherwise}) \end{cases}$$

(2)

$$Z_r^{st} = \begin{cases} \mathrm{Ker} f^{s+1} \cap \mathrm{Im}^{r-1} M^{s+1} & (s+t=-1) \\ E^{st} & (\text{otherwise}) \end{cases}$$

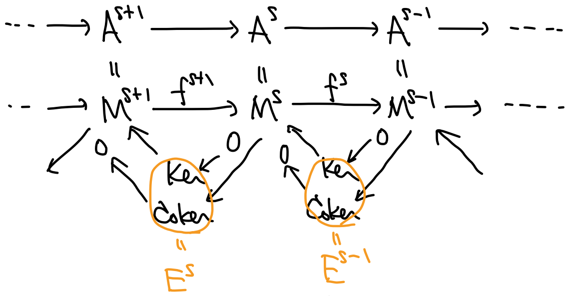
(3)

$$B_r^{st} = \begin{cases} \dfrac{K_{r-1} M^s + \mathrm{Im} f^{s+1}}{\mathrm{Im} f^{s+1}} & (s+t=0) \\ 0 & (\text{otherwise}) \end{cases}$$

proof (1) 明らか.

(2) (3)

unrolled exact couple は下図のようになっている:



Z_r^s, B_r^s の def から直接計算すれば ok //

Rmk 5.1.10

(1)より,

$$s+t \neq -1 \implies d_r^{st} = 0 \quad (\forall r)$$

よって,

$$\begin{cases} s+t \neq -1 \implies Z_r^{st} = E^{st} & (\forall r) \\ s+t \neq 0 \implies B_r^{st} = 0 & (\forall r) \end{cases}$$

(cf. Observation 5.1.8)

Def 5.1.11

$$\boxed{\mathrm{Im}^{\omega} M^s := \bigcap_r \mathrm{Im}^r M^s}$$

Rmk 5.1.12

- $\mathrm{Im}^{\omega} A^s = Q^s$ (Def 3.5.1)
- ω = (the smallest infinite ordinal)

  = (the order type of $\mathbb{N}$)

（詳細は§3.6に書いてあるけど、別に読む必要はない）

Prop 5.1.13

(1)
$$Z^{st}_{\infty} = \begin{cases} \mathrm{Ker} f^{s+1} \cap \mathrm{Im}^{\omega} M^{s+1} & (s+t=-1) \\ E^{st} & (\text{otherwise}) \end{cases}$$

(2)
$$B^{st}_{\infty} = \begin{cases} \mathrm{Coker} f^{s+1} \ (= E^{st}) & (s+t=0) \\ 0 & (\text{otherwise}) \end{cases}$$

~~proof~~ Lem 5.1.9 //

Prop 5.1.14

(1)
$$E^{st}_{\infty} = \begin{cases} \mathrm{Ker} f^{s+1} \cap \mathrm{Im}^{\omega} M^{s+1} & (s+t=-1) \\ 0 & (\text{otherwise}) \end{cases}$$

(2)
$$RE^{st}_{\infty} = \begin{cases} R\lim_r (\mathrm{Ker} f^{s+1} \cap \mathrm{Im}^{r-1} M^{s+1}) & (s+t=-1) \\ 0 & (\text{otherwise}) \end{cases}$$

~~proof~~ (1) Prop 5.1.13 (2) Lem 5.1.9 (2) //

以上 (Lem 5.1.6 ～ Prop 5.1.14) をまとめて、次を得る：

Thm 5.1.15

$\{F^s_v\}$ からつくられる unrolled exact couple や spectral sequence は 次をみたす

(1) ${}^{\forall}s < -1$, $F^s = 0$

($\rightsquigarrow$ entering differentials)

(2) $A^{-\infty} = 0$ ← in deg 0 →

$A^{\infty} = \lim_s M^s$, $RA^{\infty} = R\lim_s M^s$

(3)
$$E^{st}_{\infty} = \begin{cases} \mathrm{Ker} f^{s+1} \cap \mathrm{Im}^{\omega} M^{s+1} & (s+t=-1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$RE^{st}_{\infty} = \begin{cases} R\lim_r (\mathrm{Ker} f^{s+1} \cap \mathrm{Im}^{r-1} M^{s+1}) & (s+t=-1) \\ 0 & (\text{otherwise}) \end{cases}$$

Ex 5.1.16

$\{M^s\}_s$: as in Ex 1.2.2

(i.e. $\cdots \to \mathbb{K} \xrightarrow{\mathrm{id}} \mathbb{K} \xrightarrow{\mathrm{id}} \mathbb{K}$)

target : $A^{-\infty}$ 収束しない (see Ex 5.1.7)

target : A^{∞}

Thm 5.1.15 (1)(2) より、

$\{E_r\}_r$: conditionally convergent to $A^{\infty} = \mathbb{K}$
with entering diff

さらに、Thm 5.1.15 (3) より、

$RE_{\infty} = 0$

よって Thm 4.3.1 より、

$\{E_r\}_r$: strongly convergent to $A^{\infty} = \mathbb{K}$

(実際、Thm 5.1.15 (3) より

$$E^{st}_{\infty} = \begin{cases} \mathbb{K} & ((s,t) = (-1,0)) \\ 0 & (\text{otherwise}) \end{cases}$$

なので、

$k^{-1} \circ \varepsilon^{s+1} : F^sA^{\infty}/F^{s+1}A^{\infty} \xrightarrow{\cong} E^s_{\infty}$: isom of deg -1)

deg $k = +1$
($A^{-\infty}$ ではなく、A^{∞} が target だから)

Ex 5.1.7 は、「$A^{\infty} \neq 0$ のせいで $A^{-\infty}$ に収束しない」例であった。
$RA^{\infty} \neq 0$ の例も与えておく。

Ex 5.1.17

← Ex 3.5.21 (1) でもok

$\{M^s\}$: as in Ex 3.5.20 (1)

(i.e. $\cdots \to \mathbb{K}[x] \xrightarrow{x\times} \mathbb{K}[x] \xrightarrow{x\times} \mathbb{K}[x] \longrightarrow \mathbb{K}$, $f \longmapsto f(1)$)

$\rightsquigarrow A^{\infty} = 0$, $RA^{\infty} = \mathbb{K}[[x]]/\mathbb{K}[x] \neq 0$

また、このとき

$E^{-1,0}_{\infty} = \mathrm{Im}^{\omega} M^0 = \mathbb{K} \neq 0$

なので、$A^{-\infty}$ (=0) には収束していない。

conditionally convergent だが
weakly convergent でない例もあげておく。

Ex 5.1.18

← Ex 3.5.21 (2) でもok

$\{M^s\}$: as in Ex 3.5.20 (2)

(i.e. $\cdots \to \mathbb{K}[[x]] \xrightarrow{x\times} \mathbb{K}[[x]] \xrightarrow{x\times} \mathbb{K}[[x]] \twoheadrightarrow \mathbb{K}[[x]]/\mathbb{K}[x]$)

$\rightsquigarrow A^{\infty} = RA^{\infty} = 0$

しかし

$E^{-1,0}_{\infty} = \mathrm{Im}^{\omega} M^0 = \mathbb{K}[[x]]/\mathbb{K}[x] \neq 0$

なので、$A^{-\infty}$ (=0) には weakly convergent でない。

($RE^{0,-1}_{\infty} = R\lim_r (\mathbb{K}[x] \cap \mathrm{Im}^{r-1} M^1) = \mathbb{K}[[x]]/\mathbb{K}[x] \neq 0$ が原因)

strongly convergent だが
conditionally convergent でない例を考える

Ex 5.1.19

Define $\{M^s\}$ by :

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{K}[x] \arrow[r, "\times x"] \arrow[d, phantom, "\shortparallel"] & \mathbb{K}[x] \arrow[r, "\times x"] \arrow[d, phantom, "\shortparallel"] & \mathbb{K}[x] \arrow[d, phantom, "\shortparallel"] \\
\cdots \arrow[r] & M^2 \arrow[r] & M^1 \arrow[r] & M^0
\end{tikzcd}$$

(same M^s with Ex 5.1.17 for $s>0$)

⤳ Ex 5.1.17 と同様に

- $A^{-\infty}=0$
- $A^{\infty}=0$, $RA^{\infty}\neq 0$

⤳ NOT conditionally conv. to $A^{-\infty}$

ここで

- $E^{st}_\infty=0 \quad ({}^\forall s,t)$
- $RE^{st}_\infty=\begin{cases}\mathbb{K}[[x]]/\mathbb{K}[x] & ((s,t)=(-1,0)) \\ 0 & (\text{otherwise})\end{cases}$

[∵ $\operatorname{Ker} f^{s+1}=\begin{cases}\mathbb{K}[x] & (s=-1)\\ 0 & (s\neq -1)\end{cases}$

$\underset{\text{Thm 5.1.15}}{\rightsquigarrow} E^{st}_\infty=RE^{st}_\infty=0$ for $(s,t)\neq(-1,0)$

また、

- $\operatorname{Im}^{\omega}M^0=0$
- $\operatorname{Ker}f^0\cap\operatorname{Im}^{r-1}M^0=x^{r-1}\mathbb{K}[x]$

より、

$E^{-1,0}_\infty=0$, $RE^{-1,0}_\infty=\mathbb{K}[[x]]/\mathbb{K}[x]$]

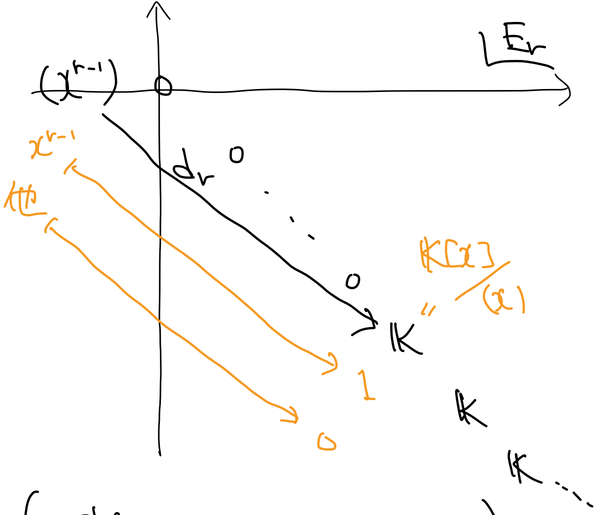
以上より、

$\{E^{st}_r\}$: strongly convergent to $A^{-\infty}$

(∵ • $\dfrac{F^s\underset{\overset{\shortparallel}{0}}{A^{-\infty}}}{F^{s+1}A^{-\infty}}\xrightarrow{\cong}\underset{\overset{\shortparallel}{0}}{E^s_\infty}$

• $A^{-\infty}=0 \rightsquigarrow \{F^sA^{-\infty}\}$: exhaustive, complete Hausdorff)

ちなみに E_r は下図のようになる:



($E^{-1,0}_r=(x^{r-1})=x^{r-1}\mathbb{K}[x]$)

Rmk 5.1.20

- Thm 4.3.5 より

  strongly conv. & $RE_\infty=0$

  $\Longrightarrow$ conditionally conv.

なので、上のような例では $RE_\infty\neq 0$ が必要

- limit を target にする場合は
  - $F^sA^{\infty}/F^{s+1}A^{\infty}\longrightarrow E^s_\infty$ を def するために $A^{-\infty}=0$ が必要 (Lem 4.1.18)
  - conditionally conv. $\underset{\text{def}}{\iff}$ $A^{-\infty}=0$

なので、上のような例は存在しない

§5.2 Filtered complexes [Boa, §9]

Fix

- C: chain cpx
- $\{F^sC\}_s$: filtration on C

Rmk 5.2.1

- 主に bounded でない場合が対象
- bounded なら Thm 1.1.10 で十分

Define

$$\begin{cases} A^s := H^*(F^sC) \\ E^s := H^*\left(F^sC/F^{s+1}C\right) \end{cases}$$

(as in §2.3)

$\rightsquigarrow$ unrolled exact couple

$\rightsquigarrow$ $\{E_r\}_r$: spectral seq

Prop 5.2.2

$$\begin{tikzcd}
d_1^{st} : & E_1^{st} \arrow[r] \arrow[d, equal] & E_1^{s+1,t} \arrow[d, equal] \\
& H^{s+t}\left(F^sC/F^{s+1}C\right) \arrow[r] & H^{s+t+1}\left(F^{s+1}C/F^{s+2}C\right) \\
& {[[x]]} \arrow[r, mapsto] & {[[dx]]}
\end{tikzcd}$$

(i.e. the connecting hom for

$$0 \to F^{s+1}C/F^{s+2}C \to F^sC/F^{s+2}C \to F^sC/F^{s+1}C \to 0$$
)

proof def からすぐに分かる (c.f. Prop 1.1.4) //

Thm 5.2.3 [Boa, Thm 9.2]

Assume

$\{F^sC\}$. exhaustive, complete, Hausdorff.

(i.e. $F^{-\infty}C = C$, $RF^{\infty}C = F^{\infty}C = 0$)

Then

$$E_1^{st} \cong H^{s+t}\left(F^sC/F^{s+1}C\right) \Rightarrow H^{s+t}(C)$$

conditionally convergent to the colimit $H^*(C)$

proof

We need to show :

$$\begin{cases} (1)\ A^{-\infty} = H^*(C) \\ (2)\ A^{\infty} = RA^{\infty} = 0 \end{cases}$$

$F^{-\infty}C = C$

(1) $A^{-\infty} = \operatorname{colim}_s H^*(F^sC) = H^*(\operatorname{colim}_s F^sC) \overset{\downarrow}{=} H^*(C)$

(2) By assump, $\leftarrow F^{\infty}C = RF^{\infty}C = 0$

$1-i : \prod_s F^sC \xrightarrow{\cong} \prod_s F^sC$: isom

$\big\downarrow H^*(-)$

$$\begin{tikzcd}
H^*(\prod_s F^sC) \arrow[r, "\cong", "H^*(1-i)"'] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & H^*(\prod_s F^sC) \arrow[d, "\cong"] \\
\prod_s \underbrace{H^*(F^sC)}_{A^s} \arrow[r, "1-H^*(i)"'] & \prod_s \underbrace{H^*(F^sC)}_{A^s}
\end{tikzcd}$$

: isom

$\leftarrow H^*$ preserves $\prod$

$\rightsquigarrow$ $A^{\infty} = RA^{\infty} = 0$

//

— 一般の filtered cpx を考えるときは…

① not exhaustive

$\rightsquigarrow$ replace C by $F^{-\infty}C$

② not complete Hausdorff

$\hat{C} := \lim_t C/F^tC$: completion of C

(see §3.4)

$F^s\hat{C} := \lim_t F^sC/F^tC$

(differential is defined by the naturality of $\lim_t$)

$\rightsquigarrow$ Thm 5.2.4

Thm 5.2.4 [Boa, Thm 9.3]

Let

$\{E_r\}$: spectral seq for $\{F^sC\}$ (as in Thm 5.2.3)

$\{\hat{E}_r\}$: ——— 〃 ——— $\{F^s\hat{C}\}$

Assume

$\{F^sC\}$: exhaustive (i.e. $F^{-\infty}C = C$)

Then

- $\hat{E}_1^{st} \cong H^{s+t}(F^s\hat{C}/F^{s+1}\hat{C}) \Longrightarrow H^{s+t}(\hat{C})$ conditionally convergent to the colimit $H^*(\hat{C})$
- $1 \leq {}^\forall r \leq \infty$, $E_r^{st} \xrightarrow{\cong} \hat{E}_r^{st}$ (compatible with d_r for $1 \leq r < \infty$)
- $RE_\infty^{st} \xrightarrow{\cong} R\hat{E}_\infty^{st}$

proof

Prop 3.4.8 より、$\{F^s\hat{C}\}_s$: exhaustive, complete Hausdorff

よって Thm 5.2.3 を $\{F^s\hat{C}\}_s$ に適用すると

$$\hat{E}_1^{st} \cong H^{s+t}(F^s\hat{C}/F^{s+1}\hat{C}) \Longrightarrow H^{s+t}(\hat{C})$$

conditionally convergent

ここで completion hom は次を induce :

$$\{F^sC\} \longrightarrow \{F^s\hat{C}\} : \text{morph of filtered cpx}$$

$$\rightsquigarrow (A^s, E^s) \longrightarrow (\hat{A}^s, \hat{E}^s) : \text{morph of unrolled exact couple}$$

Prop 3.4.8 (6) より,

$$\forall s,\ F^sC/F^{s+1}C \xrightarrow{\cong} F^s\hat{C}/F^{s+1}\hat{C}$$

$$\rightsquigarrow \forall s,\ E_1^s \xrightarrow{\cong} \hat{E}_1^s$$

$$\left(\rightsquigarrow \hat{E}_1^{st} \cong E_1^{st} \cong H^{s+t}(F^sC/F^{s+1}C) \right)$$

よって Prop 4.1.9 より

$$1 \leq {}^\forall r \leq \infty,\ E_r \xrightarrow{\cong} \hat{E}_r,\quad RE_\infty \xrightarrow{\cong} R\hat{E}_\infty$$

//

Rmk 5.2.5

一般論で言えるのは conditional convergence まで.

strong convergence は, 個々の case に応じて §4.2〜§4.4 の Thm を使う.

- ${}^\exists s \in \mathbb{Z}$, $F^sC = 0$ ← degreewise でもok

  $\Longrightarrow$ exiting differentials (§4.2)

- ${}^\exists s \in \mathbb{Z}$, $F^sC = C$ ←

  $\Longrightarrow$ entering differentials (§4.3)

§5.3 Double complexes [Boa, §10]

$\{D^{st}\}_{s,t}$: double cpx

$d_H : D^{st} \to D^{s+1,t}$, $d_V : D^{st} \to D^{s,t+1}$

Recall (from §1.2)

$$\mathrm{Tot}^n D = \bigoplus_{i+j=n} D^{ij} \qquad d = d_H + d_V$$

$$F_V^s \mathrm{Tot}^n D = \bigoplus_{\substack{i+j=n \\ i \geq s}} D^{ij}$$

For simplicity, we write

$$C := \mathrm{Tot}\, D, \quad F^s C := F_V^s \mathrm{Tot}\, D$$

Lem 5.3.1

(1) $\{F^s C\}$: exhaustive (i.e. $F^{-\infty} C = C$)

(2) $\{F^s C\}$: Hausdorff (i.e. $F^{\infty} C = 0$)

proof 明らか. //

一般には complete とは限らないので.

completion $(\hat{C}, \hat{F}^s C)$ を考える.

Lem 5.3.2

$$\hat{C}^n = \left\{ \{x^s\}_s \in \prod_{s+t=n} D^{st} \;\middle|\; \exists s_0, \forall s < s_0, x^s = 0 \right\}$$

$x^s \in D^{st}$

depends on x

proof def から直接確認するだけ //

Lem 5.3.3

degreewiseでも良い

(1) Assume $\exists s_0, \forall s < s_0, D^{st} = 0$

Then $\hat{C}^n = \prod_{s+t=n} D^{st}$

(2) Assume $\exists s_0, \forall s > s_0, D^{st} = 0$

Then $\hat{C}^n = C^n = \bigoplus_{s+t=n} D^{st}$

proof Lem 5.3.2 より明らか. //

Thm 5.3.4 [Boa, Thm 10.1]

We have a spectral seq $\{E_r\}_r$ s.t.

$$E_2 \cong H^*(H^*(D, d_V), H^*(d_H)) \Rightarrow H^*(\hat{C})$$

conditionally convergent to the colimit $H^*(\hat{C})$

proof Thm 5.2.4 より 収束は OK.

E_2 は def からすぐに計算できる. //

Ex 5.3.5

Let $\{D^{st}\}$ be the double cpx in §5.1

$$D = \left(\begin{tikzcd} M^0 \arrow[r, "id"] & M^0 \\ & M^1 \arrow[u, "-f^1"'] \arrow[r, "id"] & M^1 \cdots \end{tikzcd} \right)$$

Thm 5.3.4 を適用するとどうなるか考える.

⑴ Spectral seq for $\{F_H^t\}_t$

左の議論で s と t を入れかえたものと考える.

このとき

$\forall t > 0$, $D^{st} = 0$

なので Lem 5.3.3 より,

$\hat{C} = C$

$\rightsquigarrow$ §5.1 と何も変わらない.

⑵ Spectral seq for $\{F_V^s\}_s$

- $\forall s < -1$, $D^{st} = 0$

なので, Lem 5.3.3 (1) より,

$$\hat{C}^n = \prod_{s+t=n} D^{st} \quad (\neq C^n)$$

$$\rightsquigarrow H^n(\hat{C}) \cong \begin{cases} \lim_s M^s & (n=-1) \\ R\lim_s M^s & (n=0) \\ 0 & (\text{otherwise}) \end{cases}$$

- Thm 5.3.4 より conditionally convergent
- strong convergence には $RE_\infty = 0$ が必要.

Thm 5.2.4 より, Prop 5.1.14 (2) が使える.

e.g.

Ex 1.2.2 の場合, つまり

$$\cdots \to \mathbb{K} \xrightarrow{id} \mathbb{K} \xrightarrow{id} \mathbb{K}$$

のとき, 得られる spectral seq は

$$E_1 \cong \mathbb{K} \Rightarrow \mathbb{K} \text{ : strongly convergent}$$

となる

(☺ $\mathrm{colim}\, A^s = H^*(\hat{C}) = \mathbb{K}$)

Rmk 5.3.6

degreewiseでも良い.

(1) $\exists s_0, \forall s > s_0$, $D^{st} = 0$

$\Longrightarrow$ exiting differentials (§4.2)

(2) $\exists s_0, \forall s < s_0$, $D^{st} = 0$

$\Longrightarrow$ entering differentials (§4.3)

§5.4 Another spectral sequence for filtered complex

(essentially [Boa, §12])

Fix
- C : chain cpx
- $\{F^sC\}_s$: filtration on C

⇝ spectral sequence を2つ作って比較する

① $\begin{cases} A^s = H(F^sC) \\ E^s := H(F^sC/F^{s+1}C) \end{cases}$ as in §5.2

$$\begin{tikzcd}
\cdots \arrow[r] & H(F^{s+1}C) \arrow[r] & H(F^sC) \arrow[r] \arrow[dl] & \cdots \\
& H(F^sC/F^{s+1}C) \arrow[u, "\boxed{\deg +1}"] & &
\end{tikzcd}$$

② $\begin{cases} \overline{A}^s = H(C/F^sC) \\ \overline{E}^s := H(F^sC/F^{s+1}C) \end{cases}$

$$\begin{tikzcd}
\cdots \arrow[r] & H(C/F^{s+1}C) \arrow[r] & H(C/F^sC) \arrow[r] \arrow[dl, "\boxed{\deg +1}"] & \cdots \\
& H(F^sC/F^{s+1}C) \arrow[u] & &
\end{tikzcd}$$

(where $0 \to F^sC/F^{s+1}C \to C/F^{s+1}C \to C/F^sC \to 0$)

この2つを比較したい.

Let

$\delta : \overline{A}^s \longrightarrow A^s$: deg $+1$

be the connecting hom for

$0 \to F^sC \to C \to C/F^sC \to 0$

Prop 5.4.1

(1) $(\delta, \mathrm{id}) : (\overline{A}^s, \overline{E}^s) \longrightarrow (A^s, E^s)$: morph of unrolled exact couples

(2) (δ, id) induces
- $1 \leq {}^\forall r \leq \infty$, $\overline{E}_r \xrightarrow{\cong} E_r$
- $R\overline{E}_\infty \longrightarrow RE_\infty$

proof

(1)

$$\begin{tikzcd}
\overline{E}^s \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E^s \arrow[d, "\boxed{+1}"'] \\
\overline{A}^{s+1} \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A^{s+1} \arrow[d] \\
\overline{A}^s \arrow[r, "\delta"] \arrow[d, "\boxed{+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & A^s \arrow[d] \\
\overline{E}^s \arrow[r, "\mathrm{id}"] & E^s
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & F^{s+1} \arrow[r] \arrow[d, "="] \arrow[dr, phantom, "\circlearrowleft"] & F^s \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F^s/F^{s+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{s+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "="] \arrow[dr, phantom, "\circlearrowleft"] & C/F^{s+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^s \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C/F^s \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & F^s/F^{s+1} \arrow[r] & C/F^{s+1} \arrow[r] & C/F^s \arrow[r] & 0
\end{tikzcd}$$

(2) (1) + Prop 4.1.9

(∵ $\mathrm{id} : \overline{E}^s \xrightarrow{\cong} E^s$: isom) //

よって「同じ spectral seq」となっている

収束は---？

①② それぞれについて 収束を調べる

Assume

(A1) $\forall s < 0,\quad F^sC = C$

(A2) $F^\infty C = 0$ ← entering diff

(A3) $RF^\infty C = 0$

(exiting diff は扱われないが entering diff よりも簡単なはず (「全てうまくいく」？))

① $A^s = H(F^sC)$

(こっちは §5.2で 部分的に調べた)

Prop 5.4.2 (1) だけ

Prop 5.4.2

(1) $\{E_r\}$: conditionally convergent to the colimit $A^{-\infty} = H(C)$

$\left(\begin{array}{l} \cdot\ A^{-\infty} = H(C) \\ \cdot\ A^{\infty} = RA^{\infty} = 0 \end{array}\right)$

(2) $\{F^sA^{-\infty}\}_s$: complete, exhaustive

(3) TFAE:

(i) $\{E_r\}$: strongly convergent to the colimit $A^{-\infty} = H(C)$

(ii) $\begin{cases} \text{(a) } \forall s,\ \text{“}j\circ(\eta^s)^{-1}\text{”}: F^sA^{-\infty}/F^{s+1}A^{-\infty} \xrightarrow{\cong} E^s_\infty \\ \text{(b) } \{F^sA^{-\infty}\} : \text{Hausdorff} \\ \quad (\text{i.e. } \lim_s F^sA^{-\infty} = 0) \end{cases}$

(iii) $RE_\infty = 0$

proof

(1) Thm 5.2.3 ← (A1), (A2), (A3)

(2) exhaustive 常に成立 (Lem 4.1.17)

complete Lem 4.1.26 ← (1)

(3)

(i) ⟺ (ii) (2)

(i) ⟺ (iii) (1) + Thm 4.3.5

//

後で ② と比較するために、

Lem 4.1.18 の “$j\circ(\eta^s)^{-1}$” を具体的に記述しておく

Lem 5.4.3

$$\text{“}j\circ(\eta^s)^{-1}\text{”}: F^sA^{-\infty}/F^{s+1}A^{-\infty} \longrightarrow E^s_\infty$$

$$[[x]] \longmapsto [[[x]]]$$

where

$x \in F^sC,\ dx = 0$

- LHS について：

$[x] \in \mathrm{Im}(H(F^sC) \to H(C)) = F^sA^{-\infty}$

$[[x]] \in F^sA^{-\infty}/F^{s+1}A^{-\infty}$

- RHS について：

$[x] \in F^sC/F^{s+1}C$

$[[x]] \in Z^s_\infty \subset E^s = H(F^sC/F^{s+1}C)$

$[[[x]]] \in Z^s_\infty/B^s_\infty = E^s_\infty$

proof

$$\begin{tikzcd}
F^sA^{-\infty} \arrow[d, phantom, "\shortparallel"] & A^s \arrow[l, two heads, "\eta^s"'] \arrow[r, "j"] \arrow[d, phantom, "\shortparallel"] & E^s \arrow[d, phantom, "\shortparallel"] \\
\mathrm{Im} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H(F^sC) \arrow[l, two heads] \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H(F^sC/F^{s+1}C) \\
{[x]} & {[x]} \arrow[l, mapsto] \arrow[r, mapsto] & {[[x]]}
\end{tikzcd}$$

//

② $\overline{A}^s = H(C/F^sC)$

Prop 5.4.4

(1) $\{\overline{E}_r\}$: conditionally convergent to the limit $\overline{A}^\infty = \lim_s H(C/F^sC)$

$(\overline{A}^{-\infty} = 0)$

(2) $0 \to R\overline{A}^\infty \longrightarrow H(C) \longrightarrow \overline{A}^\infty \to 0$: exact

(3) $\{F^s\overline{A}^\infty\}_s$: complete, Hausdorff, exhaustive

(4) TFAE:

(i) $\{\overline{E}_r\}$: strongly convergent to $\overline{A}^\infty$

(ii) $\forall s$, "$k^{-1}\circ\varepsilon^{s+1}$": $F^s\overline{A}^\infty / F^{s+1}\overline{A}^\infty \xrightarrow{\cong} \overline{E}^s_\infty$ —— $(\overline{a})$

(5) TFAE:

(i) $R\overline{E}_\infty = 0$

(ii) $\begin{cases} (\overline{a}) & \text{(as above)} \\ (\overline{b}) & R\overline{A}^\infty = 0 \end{cases}$

proof

(1) By (A1), $\forall s<0$ $\overline{A}^s = 0$

$\leadsto \overline{A}^{-\infty} = 0$

(2) $C/F^{s+1}C \twoheadrightarrow C/F^sC$: surj

$\leadsto R\lim_s (C/F^sC) = 0$

よって Thm 3.3.2 (2) より

$$0 \to \underset{\overset{\parallel}{\overline{A}^s}}{R\lim_s H(C/F^s)} \longrightarrow \underset{\overset{\parallel \,\nwarrow (A2)(A3)}{C}}{H(\lim_s(C/F^s))} \longrightarrow \underset{\overset{\parallel}{\overline{A}^s}}{\lim_s(H(C/F^s))} \to 0 \quad \text{exact}$$

(3) Lem 4.1.17 より

- complete Hausdorff : 常に成立
- exhaustive : $\overline{A}^{-\infty} = 0$ より ok

$\left(\operatorname{colim}_s F^s\overline{A}^\infty = \operatorname{Ker}(\overline{A}^\infty \to \overset{\overset{0}{\parallel}}{\overline{A}^{-\infty}}) \right)$

(4) (3)より ok

(5) Thm 4.3.6 より ok

//

こっちでも map を記述しておく：

Lem 5.4.5

$$\text{"}k^{-1}\circ\varepsilon^{s+1}\text{"} : F^s\overline{A}^\infty / F^{s+1}\overline{A}^\infty \longrightarrow \overline{E}^s_\infty$$

$$[\{[[x^t]]\}_t] \longmapsto [[[x^{s+1} - dy]]]$$

where

- $\{\underset{\overset{\cap}{H(C/F^tC)}}{[[x^t]]}\}_t \in F^s\overline{A}^\infty \subset \overline{A}^\infty = \lim_t H(C/F^tC)$

$\left(x^t \in C,\ dx^t \in F^tC,\ \text{conditions for } F^s\overline{A}^\infty \right)$

- $y \in C$ s.t. $x^{s+1} - dy \in F^sC$
- $[[x^{s+1} - dy]] \in \overline{Z}^s_\infty \subset \overline{E}^s = H(F^sC/F^{s+1}C)$

$[[[x^{s+1} - dy]]] \in \overline{Z}^s_\infty / B^s_\infty = \overline{E}^s_\infty$

proof

$$\begin{tikzcd}
F^s\overline{A}^\infty \arrow[r, hook] \arrow[dd, phantom, "\ni"] & \overline{A}^\infty \arrow[r, "\varepsilon^{s+1}"] & \overline{A}^{s+1} \arrow[d, phantom, "\parallel"] & \overline{E}^s \arrow[l, "k"'] \arrow[d, phantom, "\parallel"] \\
 & & H(C/F^{s+1}) \arrow[d, phantom, "\ni"] & H(F^s/F^{s+1}) \arrow[l] \arrow[d, phantom, "\ni"] \\
\{[[x^t]]\}_t \arrow[rr, mapsto] & & {[[x^{s+1}]]} & {[[x^{s+1} - dy]]} \arrow[l, mapsto]
\end{tikzcd}$$

Since

$\{[[x^t]]\}_t \in F^s\overline{A}^\infty = \operatorname{Ker}\left(\lim_t(H(C/F^t)) \longrightarrow H(C/F^s)\right)$

we have

$[[x^{s+1}]] = 0 \in H(C/F^s)$

$\leadsto \exists y \in C$ s.t. $[x^{s+1}] = d[y] \in C/F^s$

i.e. $x^{s+1} - dy \in F^s$

$\leadsto [[x^{s+1} - dy]] \in H(F^s/F^{s+1})$

$\left(\because\ d(x^{s+1} - dy) = dx^{s+1} \underset{\uparrow \left([[x^{s+1}]] \in H(C/F^{s+1})\right)}{\in} F^{s+1} \right)$

(このような元は勝手に

$[[x^{s+1} - dy]] \in \overline{Z}^s_\infty$

となることに注意

(see the proof of Lem 4.1.18))

//

• 2つの比較

$\{E_r\}$ と $\{\bar{E}_r\}$ の収束を比較したい。

$A^{-\infty} = H(C)$ なので、Prop 5.4.4 (2) より

$$\begin{tikzcd}
0 \arrow[r] & R\bar{A}^{\infty} \arrow[r] & A^{-\infty} \arrow[r] \arrow[d, phantom, "\cup"] & \bar{A}^{\infty} \arrow[r] \arrow[d, phantom, "\cup"] & 0 & \text{: exact} \\
 & & F^sA^{-\infty} & F^s\bar{A}^{\infty} & & (*)
\end{tikzcd}$$

精密化する：

Prop 5.4.6

$$0 \to R\bar{A}^{\infty} \to F^sA^{-\infty} \xrightarrow{\varphi^s} F^s\bar{A}^{\infty} \to 0 \text{ : exact}$$

(s について natural)

(where

$$\begin{tikzcd}
\varphi^s : F^sA^{-\infty} \arrow[rr] \arrow[d, phantom, "\cap"] & & F^s\bar{A}^{\infty} \arrow[d, phantom, "\cap"] \\
H(C) \arrow[rr] & & \varprojlim_t H(C/F^t) \\
{[x]} \arrow[rr, mapsto] & & \{[[x]]\}_t
\end{tikzcd}$$
)

証明の準備：

Lem 5.4.7

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"] & 0 & \text{: exact} \\
 & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & & \text{: exact}
\end{tikzcd}$$

Then

$$0 \to A \xrightarrow{f} \beta^{-1}(\mathrm{Im} f') \xrightarrow{g} \mathrm{Ker}\,\gamma \to 0$$
exact

proof

仮定より次図を得る：

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"] & 0 & \text{: exact} \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Coker} f' \arrow[r] & C' & & \text{: exact}
\end{tikzcd}$$

ここの可換性にαの存在が必要

上図にsnake lemmaを使うと、Kerの段が欲しいものになっている

//

proof of Prop 5.4.6

By (*), we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & R\bar{A}^{\infty} \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\circlearrowleft"] & A^{-\infty} \arrow[r] \arrow[d, "="] \arrow[dr, phantom, "\circlearrowleft"] & \bar{A}^{\infty} \arrow[r] \arrow[d, "\varepsilon^s"] & 0 & \text{: exact} \\
0 \arrow[r] & F^sA^{-\infty} \arrow[r] & A^{-\infty} \arrow[r] & \bar{A}^s & & \text{: exact}
\end{tikzcd}$$

[∵ 右の可換性は明らか

$$\left(\begin{tikzcd}
H(C) \arrow[r] \arrow[d, "="] \arrow[dr, phantom, "\circlearrowleft"] & \varprojlim_t H(C/F^t) \arrow[d] \\
H(C) \arrow[r] & H(C/F^s)
\end{tikzcd}\right)$$

あとは下段のexactnessを示せばよい

$$0 \to F^s \to C \to C/F^s \to 0 \text{ : exact}$$

$$\rightsquigarrow H(F^s) \to HC \to H(C/F^s) \text{ : exact}$$

$$\begin{tikzcd}
\rightsquigarrow 0 \arrow[r] & \mathrm{Im}(HF^s \to HC) \arrow[r] \arrow[d, phantom, "\parallel"] & HC \arrow[r] \arrow[d, phantom, "\parallel"] & H(C/F^s) \arrow[d, phantom, "\parallel"] \\
 & F^sA^{-\infty} & A^{-\infty} & \bar{A}^s \quad \text{exact}
\end{tikzcd}$$

(点線は Ker の universality から誘導される)]

$\rightsquigarrow$ Lem 5.4.7 より ok

//

Cor 5.4.8

φ^s induces

$$\varphi^s : F^sA^{-\infty}/F^{s+1}A^{-\infty} \xrightarrow{\cong} F^s\bar{A}^{\infty}/F^{s+1}\bar{A}^{\infty}$$

isom

proof

$$\begin{tikzcd}
0 \arrow[r] & R\bar{A}^{\infty} \arrow[r] \arrow[d, "="] \arrow[dr, phantom, "\circlearrowleft"] & F^{s+1}A^{-\infty} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F^{s+1}\bar{A}^{\infty} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R\bar{A}^{\infty} \arrow[r] & F^sA^{-\infty} \arrow[r] & F^s\bar{A}^{\infty} \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow$ snake lemma

//

Cor 5.4.9

$$F^\infty A^{-\infty} \cong R\overline{A}^\infty$$

proof

Apply $\lim_s$ to Prop 5.4.6 //

Cor 5.4.10

(b) $F^\infty A^{-\infty} = 0$ $\iff$ $(\overline{b})$ $R\overline{A}^\infty = 0$

(in Prop 5.4.2) (in Prop 5.4.4)

proof Cor 5.4.9 より ok //

Prop 5.4.11

$$\begin{tikzcd}
F^sA^{-\infty}/F^{s+1}A^{-\infty} \arrow[r, "\cong", "\varphi^s"'] \arrow[d, "j\circ(\eta^s)^{-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & F^s\overline{A}^\infty/F^{s+1}\overline{A}^\infty \arrow[d, "k^{-1}\circ\varepsilon^{s+1}"] \\
E^s_\infty \arrow[r, "="] & \overline{E}^s_\infty
\end{tikzcd}$$

commutative

proof Lem 5.4.3, Lem 5.4.5 より

$x \in F^sC$

$$\begin{tikzcd}
F^sA^{-\infty} \arrow[r, "\varphi^s"] \arrow[dd] & F^s\overline{A}^\infty \arrow[dd] \\
{[x]} \arrow[r, mapsto] \arrow[dd, "\text{Lem5.4.3}"] & {\{[[x]]\}_t} \arrow[dd] \\
E^s_\infty \arrow[r, "="] & \overline{E}^s_\infty \\
{[[[x]]]} \arrow[r, mapsto] & {[[[x]]]}
\end{tikzcd}$$

(Lem 5.4.5 $x \in F^sC$ なので $y=0$ とできる)

//

Cor 5.4.12

(a) $F^sA^{-\infty}/F^{s+1}A^{-\infty} \cong E^s_\infty$ $\iff$ $(\overline{a})$ $F^s\overline{A}^\infty/F^{s+1}\overline{A}^\infty \cong \overline{E}^s_\infty$

(in Prop 5.4.2) (in Prop 5.4.4)

proof Prop 5.4.11 より ok //

以上をまとめて、次を得る:

Thm 5.4.13

(1)
- $1 \leq {}^\forall r \leq \infty$, $\overline{E}_r \xrightarrow{\cong} E_r$
- $R\overline{E}_\infty \xrightarrow{\cong} RE_\infty$

(A1)〜(A3) を仮定すると、次も成立:

(2) TFAE:

(i) $\{E_r\}_r$: weakly convergent to $A^{-\infty}$

(ii) $\{\overline{E}_r\}_r$: weakly convergent to $\overline{A}^\infty$

(iii) $\{\overline{E}_r\}_r$: strongly convergent to $\overline{A}^\infty$

(3) TFAE:

(i) $\{E_r\}_r$: strongly convergent to $A^{-\infty}$

(ii) $\{\overline{E}_r\}_r$: strongly convergent to $\overline{A}^\infty$ and $R\overline{A}^\infty = 0$

また、このとき次が成立:
- $A^{-\infty} \xrightarrow{\cong} \overline{A}^\infty$: isom of filtered modules
- $RE_\infty = R\overline{E}_\infty = 0$

proof (1) Prop 5.4.1 の再掲

(2)(3)

Prop 5.4.2, Prop 5.4.4, Prop 5.4.6, Cor 5.4.10, Cor 5.4.12 をまとめた.

((2) は (a), $(\overline{a})$ に対応

(3) は $\begin{cases} \text{(a) and (b)} \\ (\overline{a}) \text{ and } (\overline{b}) \end{cases}$ に対応する.)

//

Rmk 5.4.14

(2) の case だと, $A^{-\infty} \xrightarrow{\cong} \overline{A}^\infty$ とは限らない

$\left(0 \to \underset{\neq 0}{R\overline{A}^\infty} \longrightarrow A^{-\infty} \longrightarrow \overline{A}^\infty \to 0 \text{, exact} \right)$

しかし, weak convergence は同値になっている

§5.5 Filtrations on ⊗ and Hom [NoRef]

Notation

$\mathbb{K}$: commutative ring

$\otimes := \otimes_{\mathbb{K}}$, $\mathrm{Hom} := \mathrm{Hom}_{\mathbb{K}}$

Aim

C, D : cpx $/\mathbb{K}$

C or D : filtered

$\rightsquigarrow$ $C \otimes D$, $\mathrm{Hom}(C, D)$: filtered

complete? Hausdorff? exhaustive?

exactness を保つために splitting を仮定する

(flat とかでも良いけど)

Lem 5.5.1

M : (graded) module

$M \supset F^0 \supset F^1$: submodules

(1) $F^1 \hookrightarrow M$: split $\Rightarrow$ $F^1 \hookrightarrow F^0$: split

(2) $F^0 \hookrightarrow M$: split $\Rightarrow$ $F^0/F^1 \hookrightarrow M/F^1$: split

proof 前者の retraction を制限すれば

後者の ——〃—— が得られる //

Def 5.5.2

M : (graded) module

$\{F^sM\}_s$: filtration on M

Then

$\{F^sM\}$: good filtration

$\overset{\text{def}}{\iff}$ $\forall s$, $F^sM \hookrightarrow M$: split

- ここだけの用語
- Lem 5.5.1 より、この仮定で十分

● Filtration on ⊗

Def 5.5.3

C, D : cpx $/\mathbb{K}$

- Define

$C \otimes D$: cpx

by

$$\begin{cases} (C \otimes D)^n := \bigoplus_{p+q=n} C^p \otimes D^q \\ d(x \otimes y) := dx \otimes y + (-1)^{|x|} x \otimes dy \end{cases}$$

- $\{F^sC\}$: good filtration on C ← filtration as cpx, split as module

Define

$F^s(C \otimes D) := F^sC \otimes D$

$\subset C \otimes D$

$F^sC \hookrightarrow C$: split

Prop 5.5.4

$C, D, \{F^sC\}$: as above

$K := C \otimes D$

Then

(1) $F^{-\infty}C = C \Rightarrow F^{-\infty}K = K$

(2) $\exists s_0, F^{s_0}C = 0 \Rightarrow \exists s_0, F^{s_0}K = 0$

($\rightsquigarrow F^{\infty}K = RF^{\infty}K = 0$)

(2') $\begin{cases} \cdot\ \forall n, \exists s_0(n), F^{s_0(n)}C^n = 0 \\ \cdot\ C = C^{\geq 0}, D = D^{\geq 0} \\ \quad (\text{i.e. } \forall n < 0, C^n = 0) \end{cases}$

$\Longrightarrow \forall n, \exists s_1(n), F^{s_1(n)}K^n = 0$

($\rightsquigarrow F^{\infty}K = RF^{\infty}K = 0$)

proof (1) colim と ⊗ の可換性

(2) $F^{s_0}K = F^{s_0}C \otimes D = 0$

(2') $s_1(n) := \max\{s_0(0), s_0(1), \dots, s_0(n)\}$ とdef.

$$F^{s_1(n)}K^n = \bigoplus_{p+q=n} \underbrace{F^{s_1(n)}C^p}_{\overset{\shortparallel}{0}} \otimes D^q = 0$$

$\rightsquigarrow$ $0 \leq p \leq n$

finite sum

($\because$ $s_1(n) \geq s_0(p)$) //

Rmk 5.5.5

(2') の仮定について : $C = C^{\geq p_0}$, $D = D^{\geq q_0}$

- bounded なら 0 じゃなくても良
- $C = C^{\leq 0}$, $D = D^{\leq 0}$ でも良

($C = C^{\leq 0}$, $D = D^{\geq 0}$ とかはダメ)

Thm 5.5.6

C, D : cpx / k

$\{F^sC\}$: good filtration on C

$K := C \otimes D$, $F^sK := F^sC \otimes D$

Assume

- $F^{-\infty}C = C$
- $\exists s_0, F^{s_0}C = 0$

Then

The spectral sequence for $\{F^sK\}$ satisfies :

(1) s.s. with exiting diff.

(2) $E_1^s = H^*\left(F^sC/F^{s+1}C \otimes D\right)$

(3) strongly convergent to the colimit $H^*(K)$

~~proof~~ (1) 明らか

(2) $0 \to F^{s+1}C \to F^sC \to F^sC/F^{s+1}C \to 0$ split exact

Lem 5.5.1(1)

$$\begin{tikzcd} {} \arrow[d, squiggly, "\otimes D"] \\ {} \end{tikzcd}$$

$0 \to F^{s+1}K \to F^sK \to F^sC/F^{s+1}C \otimes D \to 0$

(split) exact

(3) Prop 5.5.4(2) より $A^\infty = 0$

$\xrightarrow[\text{thm 4.2.1}]{}$ strongly convergent to $A^{-\infty} = H^*(K)$

//

◎ Filtration on Hom

Def 5.5.7

C, D : cpx / k

- Define

$\mathrm{Hom}(C, D)$: cpx

by

$$\begin{cases} \mathrm{Hom}^n(C, D) := \prod_{q-p=n} \mathrm{Hom}(C^p, D^q) \\ d(f) := d \circ f - (-1)^{|f|} f \circ d \quad (= [d, f]) \end{cases}$$

- $\{F^sC\}$: good filtration

Define

$$F^s\mathrm{Hom}(C, D) := \mathrm{Hom}\left(C/F^{-s+1}C, D\right) \subset \mathrm{Hom}(C, D)$$

$$\underline{F}^s\mathrm{Hom}(C, D) := \mathrm{Hom}(F^{-s+1}C, D) \twoheadleftarrow \mathrm{Hom}(C, D)$$

$F^{-s+1}C \hookrightarrow C$: split

Lem 5.5.8

$C, D, \{F^sC\}$: as above

$K := \mathrm{Hom}(C, D)$

(1) $0 \to F^{s+1}K \to F^sK \to \mathrm{Hom}\left(F^{-s}C/F^{-s+1}C, D\right) \to 0$

(split) exact

(2) $\underline{F}^sK = K/F^sK$

~~proof~~

(1) $0 \to F^{-s}/F^{-s+1} \to C/F^{-s+1} \to C/F^{-s} \to 0$

split exact

Lem 5.5.1(2)

Apply $\mathrm{Hom}(-, D)$

(2) $0 \to F^{-s+1} \to C \to C/F^{-s+1} \to 0$

split exact

Apply $\mathrm{Hom}(-, D)$

//

Rmk 5.5.9

§5.4の記号で

$$\begin{cases} \cdot\ \{F^sK\} \rightsquigarrow (A^s, E^s) \\ \cdot\ \{\underline{F}^sK\} \rightsquigarrow (\bar{A}^s, \bar{E}^s) \end{cases}$$

であり、この2つはほぼ同じs.s.を与えることを示したので、

以下では$\{F^sK\}$だけ調べる

(A1)〜(A3)は、$\{F^sK\}$の収束のために必要になる

(see Lem 5.5.10)

Prop 5.5.10

$C, D, \{F^sC\}, K, \{F^sK\}$: as above

Then

(1) $\exists s_0,\ F^{s_0}C = 0 \Rightarrow \exists s_0',\ F^{s_0'}K = K$

$(\rightsquigarrow F^{-\infty}K = K)$

(1') $\begin{cases} \cdot\ \forall n,\ \exists s_0(n),\ F^{s_0(n)}C^n = 0 \\ \cdot\ C = C^{\leq 0},\ D = D^{\geq 0} \end{cases}$

$\Rightarrow \forall n,\ \exists s_1(n),\ F^{s_1(n)}K^n = K$

$(\rightsquigarrow F^{-\infty}K = K)$

(2) $F^{-\infty}C = C \Rightarrow F^{\infty}K = RF^{\infty}K = 0$

proof (1) $F^{-s_0+1}K = \mathrm{Hom}(C/\underset{0}{\cancel{F^{s_0}C}}, D) = K$

(1') $s_1(n) := -\max\{s_0(-n), s_0(-n+1), \dots, s_0(0)\} + 1$

$$F^{s_1(n)}K^n = \prod_{q-p=n} \mathrm{Hom}\left(C^p/\underbrace{F^{-s_1(n)+1}C^p}_{\overset{\shortparallel}{0}}, D^q\right) = 0$$

$\hookrightarrow -n \leq p \leq 0$ $\quad$ $(\because\ -s_1(n)+1 \geq s_0(p))$

(2) $F^{\infty}C = C$ より,

$\mathrm{colim}_s\, C/F^sC = 0$

$\rightsquigarrow 1-i : \bigoplus_s C/F^{-s+1}C \xrightarrow{\cong} \bigoplus_s C/F^{-s+1}C$: isom

$\xrightarrow[\mathrm{Hom}(-,D)]{} 1-i : \prod_s F^sK \xrightarrow{\cong} \prod_s F^sK$: isom

$\left(\begin{aligned} \because\ &\mathrm{Hom}(\textstyle\bigoplus_s C/F^{-s+1}C, D) \\ &= \textstyle\prod_s \mathrm{Hom}(C/F^{-s+1}C, D) = \prod_s F^sK \end{aligned}\right)$

//

Rmk 5.5.11

(1') の仮定について:

$\begin{cases} \cdot\ \text{bounded なら 0 じゃなくても ok} \\ \cdot\ C = C^{\geq 0},\ D = D^{\leq 0} \text{ でも ok} \end{cases}$

$(C = C^{\geq 0}, D = D^{\geq 0}$ とかはダメ$)$

Thm 5.5.12

C, D: cpx $/\Bbbk$

$\{F^sC\}$: good filtration on C

$K := \mathrm{Hom}(C, D)$, $F^sK := \mathrm{Hom}(C/F^{-s+1}C, D)$

Assume

- $F^{-\infty}C = C$
- $\exists s_0,\ F^{s_0}C = 0$

Then

The spectral sequence for $\{F^sK\}$ satisfies:

(1) s.s. with entering diff.

(2) $E_1^s = H^*(\mathrm{Hom}(F^{-s}C/F^{-s+1}C, D))$

(3) conditionally convergent to the colimit $H^*(K)$

proof (1) 明らか.

(2) Lem 5.5.8 (1)

(3) Prop 5.5.10 より, $F^{-\infty}K = K$, $F^{\infty}K = RF^{\infty}K = 0$

$\xrightarrow[\text{Thm 5.2.3}]{}$ cond. conv. to $A^{\infty} = H^*(K)$

//

Rmk 5.5.13

- Thm 5.5.6 と Thm 5.5.12 では, $\{F^sC\}$ についての仮定は同じもの (degreewise にすると違ってくるけど)
- とくに,「projective resolution の類」は この仮定を満たしている (e.g. semifree resol. in §5.6)

◎ Another filtration on Hom

Dの方からfiltrationを入れる

Def 5.5.14

C, D : cpx$/\Bbbk$, $K := \mathrm{Hom}(C, D)$

$\{F^sD\}$: good filtration

Define

$$F^sK := \mathrm{Hom}(C, F^sD) \subset K$$

Prop 5.5.15

$C, D, \{F^sD\}, K, \{F^sK\}$: as above

Then

(1) $\exists s_0,\ F^{s_0}D = D \Rightarrow \exists s_0,\ F^{s_0}K = K$

$(\rightsquigarrow F^{-\infty}K = K)$

(1') $\begin{cases} \cdot\ \forall n,\ \exists s_0(n),\ F^{s_0(n)}D^n = D \\ \cdot\ C = C^{\leq 0},\ D = D^{\geq 0} \end{cases}$

$\Rightarrow \forall n,\ \exists s_1(n),\ F^{s_1(n)}K^n = K^n$

$(\rightsquigarrow F^{-\infty}K = K)$

(2) $F^\infty D = 0 \Rightarrow F^\infty K = 0$

(3) $F^\infty D = RF^\infty D = 0 \Rightarrow RF^\infty K = 0$

~~proof~~ (1) $F^{s_0}K = \mathrm{Hom}(C, F^{s_0}D) = K$

(1') $s_1(n) := \max\{s_0(0), s_0(1), \dots, s_0(n)\}$

$$F^{s_1(n)}K^n = \prod_{q-p=n} \mathrm{Hom}(C^p, \underbrace{F^{s_1(n)}D^q}_{= D^q}) = K^n$$

$\rightsquigarrow 0 \leq q \leq n$ $\quad$ $(\because s_1(n) \geq s_0(q))$

(2) $\mathrm{Hom}(C,-)$ と lim の可換性

(3) $1-i : \prod_s F^sD \xrightarrow{\cong} \prod_s F^sD$: isom

$\xrightarrow[\mathrm{Hom}(C,-)]{}$ $1-i : \prod_s F^sK \xrightarrow{\cong} \prod_s F^sK$: isom

//

Rmk 5.5.16

(1') の仮定について:

$\begin{cases} \cdot \text{ bounded なら } 0 \text{ じゃなくても ok} \\ \cdot\ C = C^{\geq 0},\ D = D^{\leq 0} \text{ でも ok} \end{cases}$

Thm 5.5.17

C, D : cpx$/\Bbbk$

$\{F^sD\}$: good filtration on D

$K := \mathrm{Hom}(C, D)$, $F^sK := \mathrm{Hom}(C, F^sD)$

Assume

- $\exists s_0,\ F^{s_0}D = D$
- $F^\infty D = RF^\infty D = 0$

Then

The spectral sequence for $\{F^sK\}$ satisfies:

(1) S.S. with entering diff.

(2) $E_1^s = H^*(\mathrm{Hom}(C, F^sD/F^{s+1}D))$

(3) conditionally convergent to the colimit $H^*(K)$

~~Proof~~ (1) 明らか

(2) $0 \to F^{s+1}D \to F^sD \to F^sD/F^{s+1}D \to 0$ $\quad$ $\underset{\text{Lem 5.5.1(1)}}{\text{split}}$ exact

$\downarrow \mathrm{Hom}(C,-)$

$0 \to F^{s+1}K \to F^sK \to \mathrm{Hom}(C, F^sD/F^{s+1}D) \to 0$

(split) exact

(3) Prop 5.5.15より, $F^{-\infty}K = K$, $F^\infty K = RF^\infty K = 0$

$\xrightarrow[\text{Thm 5.2.3}]{}$ cond. conv. to $A^{-\infty} = H^*(K)$

//

Rmk 5.5.18

- Thm 5.5.17 での $\{F^sD\}$ についての仮定は, 以前の2つ (Thm 5.5.6 と Thm 5.5.12) とは異なっている
- Thm 5.5.18 の仮定は, 「injective resolution の類」が満たすもの

§5.6 Eilenberg-Moore spectral sequence

$\to$ [FHT, §20 (d)]

収束の議論は書いてないけど

Introduction

Thm 5.6.1 (Eilenberg-Moore, 66)

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

"good" pullback diagram of topological spaces

Then

$$\mathrm{Tor}^n_{\underset{\sim}{C^*(B)}}(C^*(A), C^*(E)) \xrightarrow[\text{isom}]{\cong} H^n(f^*E)$$

dga

where
- $C^*(-)$: singular cochain algebra
- $C^*(B) \xrightarrow{p^*} C^*(E)$: dga hom
  $\rightsquigarrow C^*(E)$: $C^*(B)$-module
  (similar for $C^*(A)$)

$\rightsquigarrow$ We want to compute Tor over dga

$\rightsquigarrow$ Eilenberg-Moore spectral sequence

Review on homological algebra over dga

(see [FHT, §6] for details)

$\mathbb{K}$: comm. ring, $\otimes := \otimes_{\mathbb{K}}$, $\mathrm{Hom} := \mathrm{Hom}_{\mathbb{K}}$

Def 5.6.2

- (R, d) : dga (differential graded algebra)

$\underset{\text{def}}{\iff}$
- $R = \{R^n\}_{n\in\mathbb{Z}}$: graded algebra
  (i.e. $R^p \otimes R^q \to R^{p+q}$ unital, associative)
- $d : R \to R$: $\mathbb{K}$-linear map of deg 1
  s.t. $\begin{cases} d(xy) = dx \cdot y + (-1)^{|x|} x \cdot dy \\ d \circ d = 0 \end{cases}$

- (M, d) : (R, d)-module

$\underset{\text{def}}{\iff}$
- $M = \{M^n\}_{n\in\mathbb{Z}}$: R-module
  (i.e. $R \otimes M \to M$ unital, associative)
- $d : M \to M$: $\underset{\sim}{\mathbb{K}}$-linear map of deg 1
  s.t. $\begin{cases} d(xm) = dx \cdot m + (-1)^{|x|} x \cdot dm \\ d \circ d = 0 \end{cases}$

以下 (R, d) : dga とする.

Def 5.6.3

- $\begin{cases} (M, d) : \text{left } (R,d)\text{-mod} \\ (N, d) : \text{right } (R,d)\text{-mod} \end{cases}$ に対し,

  $$(N, d) \underset{(R,d)}{\otimes} (M, d) := N \otimes M / (nx \otimes m - n \otimes xm)$$

  chain cpx / $\mathbb{K}$

- $(M, d), (N, d)$: left (R, d)-mod に対し,
  - $f : M \to N$: R-linear map of deg k
    $\underset{\text{def}}{\iff}$ ${}^\forall x \in R, {}^\forall m \in M,$
    $f(xm) = (-1)^{|x|} x f(m)$
  - $\mathrm{Hom}_{(R,d)}((M, d), (N, d))$
    $:= \{f : M \to N : R\text{-linear}\}$
    $\subset \mathrm{Hom}((M, d), (N, d))$
    sub cpx

$\leftarrow$ d は適宜省略する

Def 5.6.4

$f, g : (M, d) \to (N, d)$: chain map / R of deg 0 に対し,

$f \simeq g$: homotopic / (R, d)

$\underset{\text{def}}{\iff}$ ${}^\exists h \in \mathrm{Hom}^{-1}_R(M, N)$

s.t. $f - g = d(h)$
$(:= d \circ h + h \circ d)$

Tor, Ext を def するために、
(R,d)-mod の resolution を def する：

Def 5.6.5

- (P,d) : (R,d)-module に対し、

  (P,d) : semifree/(R,d)

  $\overset{\Longleftrightarrow}{\text{def}}$ ∃ filtration/(R,d) ← increasing なので添字は下に書く

  $0=F_{-1}P \subset F_0P \subset F_1P \subset \cdots \subset P$

  s.t.
  - $\bigcup_s F_sP = P$
  - $\forall k, \left(F_sP/F_{s-1}P, d\right)$: (R,d)-free

  (i.e. ∃V(s) : free graded 𝕜-mod s.t. $\left(F_sP/F_{s-1}P, d\right) \cong (R,d)\otimes(V(s),0)$)

- (M,d) : (R,d)-mod に対し、

  $\eta : (P,d) \longrightarrow (M,d)$

  semifree resolution/(R,d)

  $\overset{\Longleftrightarrow}{\text{def}}$
  - (P,d) : semifree/(R,d)
  - $\eta : (P,d) \xrightarrow{\simeq q} (M,d)$ quasi-isom/(R,d)

Lem 5.6.6 [FHT, Prop 6.4]

$$\begin{tikzcd} & M \arrow[d, "f"', "\simeq q"] \\ P \arrow[ru, dashed, "\exists\tilde{g}"] \arrow[r, "g"'] \arrow[ru, phantom, "\simeq_h"', near end] & N \end{tikzcd}$$

diagram of (R,d)-modules
(f, g : chain map/R)

Assume
- f : quasi-isom
- (P,d) : semifree/(R,d)

Then

∃$\tilde{g}$: (P,d) ⟶ (N,d) : chain map/R
s.t. $f\circ\tilde{g} \simeq g$

Such $\tilde{g}$ is unique up to homotopy

Prop 5.6.7 [FHT, Prop 6.6]

∀(M,d) : (R,d)-mod

∃η : ∃$(P,d) \xrightarrow{\simeq q} (M,d)$: semifree resol over (R,d)

unique up to homotopy

sketch of proof ← see Cor 5.6.17

existence 頑張って作る
(普通の free resol. とかと似た方法)

uniqueness Lem 5.6.6 //

Def 5.6.8

- (M,d) : left (R,d)-mod
  (N,d) : right (R,d)-mod に対し、

  $$\mathrm{Tor}^k_{(R,d)}\big((N,d),(M,d)\big) := H^k\big((N,d)\otimes_{(R,d)}(P,d)\big)$$

  (where $\eta : (P,d) \xrightarrow{\simeq q} (M,d)$: semifree resol/(R,d))

- (M,d), (N,d) : (R,d)-mod に対し、

  $$\mathrm{Ext}^k_{(R,d)}\big((M,d),(N,d)\big) := H^k\big(\mathrm{Hom}_{(R,d)}((Q,d),(N,d))\big)$$

  (where $\eta : (Q,d) \xrightarrow{\simeq q} (M,d)$: semifree resol/(R,d))

Rmk 5.6.9

- Prop 5.6.7 より、Tor, Ext : well-def'd
- Tor の def では (M,d) の代わりに (N,d) の resol をとっても良いし、両方の resol をとっても良い。
- 一般には bigrading は入らない。

  d = 0 on R, M, N (i.e. graded module / graded alg) の場合であれば、普通の proj. resol を使うと入る

  $\rightsquigarrow \mathrm{Tor}^k \cong \bigoplus_{p+q=k} \mathrm{Tor}^{p,q}, \mathrm{Ext}^k \cong \prod_{p+q=k} \mathrm{Ext}^{p,q}$

◎ Eilenberg-Moore resolution

Idea

$$\mathrm{Tor}_R(N,M) = H^*(N \otimes_R P)$$

(**where** $\eta : (P,d) \xrightarrow{\simeq} (M,d)$ semifree resol$/(R,d)$)

$\rightsquigarrow \{N \otimes_R F_sP\}_s$: filtration of $N \otimes_R P$

(∵ $F_{s-1}P \hookrightarrow F_sP$: split $/R$)

$\rightsquigarrow$ spectral sequence

まずは semifree resol について調べる

Lem 5.6.10

$\eta : (P,d) \longrightarrow (M,d)$: chain map$/\mathbb{K}$

$0 = F_{-1}P \subset F_0P \subset F_1P \subset \cdots \subset P$: filtration

Then

connecting hom ($= d^1$ for s.s.)

$$\cdots \xrightarrow{\partial} H\left(F_2P/F_1P\right) \xrightarrow{\partial} H\left(F_1P/F_0P\right) \xrightarrow{\partial} H(F_0P) \xrightarrow{H\eta} HM \to 0 \quad \circledast$$

chain cpx of graded mod$/\mathbb{K}$

(i.e. $\partial \circ \partial = 0$, $H\eta \circ \partial = 0$)

proof

$S \geq 2$

$$H\left(F_sP/F_{s-1}P\right) \xrightarrow{\partial} H\left(F_{s-1}P/F_{s-2}P\right) \xrightarrow{\partial} H\left(F_{s-2}P/F_{s-3}P\right)$$

$$[x] \longmapsto [dx] \longmapsto [\underbrace{ddx}_{\overset{\shortparallel}{0}}]$$

"$S = 1$"

$$H\left(F_1P/F_0P\right) \xrightarrow{\partial} H(F_0P) \xrightarrow{H\eta} HM$$

$$[x] \longmapsto [dx] \longmapsto [\underbrace{\eta dx}_{\overset{\shortparallel}{d\eta x}}] = 0$$

$\eta : \underset{\sim}{F_1}P \to M$

//

Observation 5.6.11

η : semifree resol$/(R,d)$ のとき、

$$H\left(F_sP/F_{s-1}P\right) = H\left((R,d) \otimes (V(s),0)\right)$$
$$= H(R) \otimes V(s) : \text{free}/H(R)$$

$\rightsquigarrow$ If $\circledast$: exact,

$\circledast$: free resol$/H(R)$ of $H(M)$

$\circledast$ の exactness について、次が成立:

Prop 5.6.12

η, $\{F_sP\}$: as in Lem 5.6.10

Assume

$\circledast$: exact

Then

$\eta : (P,d) \xrightarrow{\simeq} (M,d)$: quasi-isom

proof (元を取って示すこともできるが、せっかくなので spectral seq を使う)

Filter M by

$$F_sM := \begin{cases} M & (s \geq 0) \\ 0 & (s < 0) \end{cases}$$

$\rightsquigarrow \eta : (P,d) \longrightarrow (M,d)$: morph of filtered cpx

$\rightsquigarrow (A,E) \longrightarrow (\bar{A},\bar{E})$: morph of unrolled exact couple

$\rightsquigarrow \{E^r_{st}\} \longrightarrow \{\bar{E}^r_{st}\}$: morph of s.s.

- $\{E^r\}$, $\{\bar{E}^r\}$ はいずれも exiting diff

  (∵ $F_sP = F_sM = 0$ for $s < 0$)

  $\rightsquigarrow$ strongly convergent to

  $$A^{-\infty} = HP, \quad \bar{A}^{-\infty} = HM$$

- $E^2 \xrightarrow{\simeq} \bar{E}^2$: isom

  (∵ $E^1_s = H\left(F_sP/F_{s-1}P\right)$, $\bar{E}^1_s = \begin{cases} HM & (s=0) \\ 0 & (s \neq 0) \end{cases}$

  $\rightsquigarrow$

$$\begin{tikzcd} E^1 : \cdots \arrow[r, "\partial"] & H\left(F_1P/F_0P\right) \arrow[r, "\partial"] \arrow[d] & H(F_0P) \arrow[r] \arrow[d, "H\eta"] & 0 \arrow[r] & \cdots \\ \bar{E}^1 : \cdots \arrow[r] & 0 \arrow[r] & HM \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

  ✪ (the vertical map $E^1 \to \bar{E}^1$)

  ✪ : quasi-isom $\Longleftrightarrow$ $\underset{\text{assump}}{\underbrace{\circledast : \text{exact}}}$)

By Thm 4.1.13,

$H\eta : HP \xrightarrow{\simeq} HM$: isom

$$\left(\overset{\shortparallel}{A^{-\infty}} \xrightarrow{\simeq} \overset{\shortparallel}{\bar{A}^{-\infty}} \right)$$

//

Rmk 5.6.13

- Prop 5.6.12 では，η が (P, d) 全体で定まっていることを使っている

  （Lem 5.6.10 では $\eta : (F_i P, d) \to (M, d)$ だけで十分だった）

- Prop 5.6.12 の逆は不成立．

  η: semifree resol だとしても

  すなわち，

  $$\eta : \text{quasi-isom} \not\Rightarrow \circledast : \text{exact}$$

  [e.g. η: quasi-isom (with $HM \neq 0$) が given とする

  このとき

  $$F_s P := F_{s-1} P$$

  と定めれば，$\circledast$: NOT exact for $\{F_s P\}$

  $$\left(☹ \cdots \longrightarrow \underbrace{H(F_0 P)}_{\overset{\shortparallel}{0}} \longrightarrow \underbrace{HM}_{\overset{\neq}{0}} \to 0 \right)$$
  ]

以上をふまえて，次を def：

Def 5.6.14

(M, d) : (R, d) - module

に対して，

$$\eta : (P, d) \longrightarrow (M, d)$$

: Eilenberg-Moore resolution$/_{(R,d)}$

$\underset{\text{def}}{\Longleftrightarrow}$
- η : semifree resol$/_{(R,d)}$
- $\circledast$: exact

（↖ η: quasi-isom は仮定しなくても Prop 5.6.12 から従う）

Lem 5.6.15

$\eta : (P, d) \xrightarrow{\simeq_q} (M, d)$: EM resol

Then

$\circledast$: free resol$/_{HR}$ of HM

proof see Observation 5.6.11 //

Prop 5.6.16 [FHT, Prop 20.11]

(M, d) : (R, d) — mod

$$\cdots \xrightarrow{\delta} HR \otimes V(1) \xrightarrow{\delta} HR \otimes V(0) \xrightarrow{\rho} HM \to 0 \quad \boxplus$$

free resol$/_{HR}$ of HM

Then

$\exists \eta : {}^{\exists}(P, d) \xrightarrow{\simeq_q} (M, d)$: EM resol$/_{(R,d)}$

s.t. $(\circledast \text{ for } (P, d)) = \boxplus$

（proof は後まわし）

Cor 5.6.17

${}^{\forall}(M, d)$: (R, d) – mod

$\exists \eta : {}^{\exists}(P, d) \xrightarrow{\simeq_q} (M, d)$: EM resol

proof

existence of free resol + Prop 5.6.16

（graded module over graded alg without differential なので．ungraded の case と同様）

//

Prop 5.6.16 の証明のために，Lem を1つ準備する

Lem 5.6.18

$\eta : (F, d) \longrightarrow (M, d)$: chain map$/\mathbb{K}$

$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_k = F$

$z \in F$ finite filtration of (F, d)

Assume

(1) $k \geq 1$

(2) $dz \in F_{k-1}$ $\left(\Leftrightarrow [z] \in F_k/F_{k-1} : \text{cycle}\right)$

(3) $\partial : H\left(F_k/F_{k-1}\right) \longrightarrow H\left(F_{k-1}/F_{k-2}\right)$

$[[z]] \longmapsto 0$

(4) $H\left(F_{k-1}/F_{k-2}\right) \xrightarrow{\partial} H\left(F_{k-2}/F_{k-3}\right) \xrightarrow{\partial} \cdots$

$\cdots \xrightarrow{\partial} H\left(F_1/F_0\right) \xrightarrow{\partial} HF_0 \xrightarrow{H\eta} HM \to 0$: exact

Then

$\exists w \in F_{k-1}$, $\exists x \in M$

s.t. $\begin{cases} \cdot\ d(z-w) = 0 \\ \cdot\ \eta(z-w) = dx \end{cases}$

$\left(\text{i.e. } H\eta[z-w] = 0 \in HM\right)$

proof

Claim 1

$0 \leq {}^{\forall}s \leq k-2$, $\exists y_s \in F_{k-1}$

s.t. $d(z - y_s) \in F_s$

（F_sではなく F_{k-1} なことに注意）

$\because$ downward induction on s.

$\underline{s = k-2}$ By assump(3), $\partial[[z]] = 0 \in H\left(F_{k-1}/F_{k-2}\right)$

$\rightsquigarrow \exists y_{k-2} \in F_{k-1}$ s.t. $[dz] = [dy_{k-2}] \in F_{k-1}/F_{k-2}$

$\rightsquigarrow d(z - y_{k-2}) \in F_{k-2}$

$\underline{0 \leq s < k-2}$ ind. hyp より

$\exists y_{s+1} \in F_{k-1}$ s.t. $d(z - y_{s+1}) \in F_{s+1}$

$$H\left(F_{s+2}/F_{s+1}\right) \xrightarrow{\partial} H\left(F_{s+1}/F_s\right) \xrightarrow{\partial} H\left(F_s/F_{s-1}\right)$$

$$\exists[[u]] \longmapsto [[d(z - y_{s+1})]] \longmapsto 0 \quad \text{exact (4)}$$

(i.e. $u \in F_{s+2} \overset{(1)\, s<k-2}{\subset} F_{k-1}$, $du \in F_{s+1}$

$\exists u' \in F_{s+1}$ s.t. $[d(z - y_{s+1})] - [du] = [du'] \in F_{s+1}/F_s$)

$\rightsquigarrow d(z \overbrace{- y_{s+1} - u - u'}^{-y_s}) \in F_s$

Claim 2

$\exists y \in F_{k-1}$ s.t. $d(z-y) = 0$

$\because$ $\underline{k=1}$ By (2)(3), $[dz] = 0 \in H(F_0)$

$\rightsquigarrow \exists y \in F_0$ s.t. $dz = dy$

$\underline{k \geq 2}$ By Claim 1,

$\exists y_0 \in F_{k-1}$ s.t. $d(z - y_0) \in F_0$

$\rightsquigarrow H\left(F_1/F_0\right) \xrightarrow{\partial} HF_0 \xrightarrow{H\eta} HM$: exact (4)

$\exists[[v]] \longmapsto [d(z-y_0)] \longmapsto 0$

$\in F_{k-1}$ 　η : def'd on F_{k-1}

(i.e. $v \in F_1 \overset{k \geq 2}{\subset} F_{k-1}$, $dv \in F_0$

$\exists v' \in F_0$ s.t. $d(z - y_0) - dv = dv'$)

$\rightsquigarrow d(z - y_0 - v - v') = 0$

Take $y \in F_{k-1}$ as in Claim 2

$\eta : F_k \longrightarrow M$ ← ηがF_k全体で定まっている必要がある

$z - y \longmapsto$ [illegible]

cycle $\rightsquigarrow$

assump (4)

Since $H\eta : HF_0 \twoheadrightarrow HM$: surj

$\exists z' \in F_0$, $\begin{cases} \cdot\ dz' = 0 \\ \cdot\ [\eta(z-y)] = [\eta z'] \in HM \end{cases}$

$\rightsquigarrow \exists x \in M$ s.t. $\eta(z-y) - \eta z' = dx$

Define

$w := y + z' \in F_{k-1}$

//

Rmk 5.6.19

S.S. を使って上の proof を解釈することができる:

- $\exists y_s \in F_{k-1}$, $d(z - y_s) \in F_s$

$\Longleftrightarrow d^{k-s}$ "$[z]$" $\in E_s^{k-s}$ is well-def'd

$\Longleftrightarrow 1 \leq {}^{\forall}r < k-s$, $d^r[z] = 0 \in E_{k-r}^r$

- assump(4) より, ${}^{\forall}r \geq 2$, $0 < {}^{\forall}p \leq k-2$, $E_p^r = 0$

$\rightsquigarrow 2 \leq {}^{\forall}r < k-s$, $d^r[z] = 0$

- $d^1[z] = \partial[z] = 0$ (assump (3))

$\rightsquigarrow E_{k-1}^r = 0$ は不要

$\rightsquigarrow$ (4) において, $H\left(F_{k-1}/F_{k-2}\right)$ での exactness は仮定しない

ちゃんとした証明にするのは大変そう

proof of Prop 5.6.16

$$\cdots \xrightarrow{\delta} HR\otimes V(1) \xrightarrow{\delta} HR\otimes V(0) \xrightarrow{\rho} HM \to 0$$

free resol$/_{HR}$

Define

$$\begin{cases} V := \bigoplus_i V(i) \\ P := R\otimes V, \quad F_sP := R\otimes \bigoplus_{i\le s} V(i) \end{cases}$$

Fix

$\{v^s_\lambda\}_{\lambda\in\Lambda_s}$: basis of $V(s)$

By induction on s, we will define

$$\underset{\sim}{\eta_s} : (F_sP, \underset{\sim}{d}) \longrightarrow (M,d) : \text{chain map}/_R$$

s.t.

(a) $(F_{s-1}P, d) \underset{\text{subcpx}}{\subset} (F_sP, d)$, $\eta_s|_{F_{s-1}P} = \eta_{s-1}$

(b) $\forall i \le s$, $d(V(i)) \subset F_{i-1}P$

$$\left(\rightsquigarrow \varphi_i : HR\otimes V(i) \xrightarrow{\cong} H\left(F_iP/F_{i-1}P\right), \quad [x]\otimes v \longmapsto [[x\otimes v]] \right)$$

(c)

$$\begin{tikzcd}
HR\otimes V(s) \arrow[r, "\delta"] \arrow[d, "\varphi_s", "\cong"'] \arrow[rd, phantom, "\circlearrowleft"] & HR\otimes V(s-1) \arrow[r, "\delta"] \arrow[d, "\varphi_{s-1}", "\cong"'] \arrow[rd, phantom, "\circlearrowleft"] & \cdots \arrow[r, "\delta"] & HR\otimes V(0) \arrow[r, "\rho"] \arrow[d, "\varphi_0", "\cong"'] \arrow[rd, phantom, "\circlearrowleft"] & HM \arrow[r] \arrow[d, "\mathrm{id}", "="'] & 0 \\
H\left(F_s/F_{s-1}\right) \arrow[r, "\partial"] & H\left(F_{s-1}/F_{s-2}\right) \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & H(F_0) \arrow[r, "H\eta_0"] & HM \arrow[r] & 0
\end{tikzcd}$$

(⇝ 上段がexactなので下段もexact)

これが構成できれば証明が終わる.

(∵ Prop 5.6.12 より $\underset{\substack{\| \\ \mathrm{colim}_s \eta_s}}{\eta} : (\underset{\substack{\| \\ \mathrm{colim}_s F_sP}}{P}, d) \xrightarrow{\simeq q} (M,d)$ quasi-isom)

$s=0$

- $d=0$ on $V(0)$

$\rightsquigarrow (F_0P, d) = (R, d) \otimes (V(0), 0)$

- Define

$$\begin{tikzcd}
\eta_0 : R\otimes V(0) \arrow[r] & M : \text{chain map}/_R \\
1\otimes v^0_\lambda \arrow[r, mapsto] \arrow[d, mapsto, "d"] & x^0_\lambda \arrow[d, mapsto, "d"] \\
0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

(where

$x^0_\lambda \in M$: cycle s.t. $[x^0_\lambda] = \rho(1\otimes v^0_\lambda) \in HM$ ——①

$\rho : HR\otimes V(0) \longrightarrow HM$

$1\otimes v^0_\lambda \longmapsto [x^0_\lambda]$)

(a)(b)は明らかにみたす.

(c)

$$\begin{tikzcd}
HR\otimes V(0) \arrow[r, "\rho"] \arrow[d, "\varphi_0"', "\cong"] & HM \arrow[r] \arrow[d, "\mathrm{id}", "="'] & 0 \\
H(F_0P) \arrow[r, "H\eta_0"] & HM \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes v^0_\lambda \arrow[r, mapsto] \arrow[d, mapsto] & \rho(1\otimes v^0_\lambda) \arrow[d, equal, "\text{①}"] \\
{[1\otimes v^0_\lambda]} \arrow[r, mapsto] & {[x^0_\lambda]}
\end{tikzcd}$$

$s=1$

Take $z^1_\lambda \in F_0P$: cycle

s.t. $[z^1_\lambda] = \varphi_0 \circ \delta(1\otimes v^1_\lambda) \in H(F_0P)$

Then we have a diagram : ——②

$$\begin{tikzcd}
HR\otimes V(1) \arrow[r, "\delta"] & HR\otimes V(0) \arrow[r, "\rho"] \arrow[d, "\varphi_0", "\cong"'] \arrow[rd, phantom, "\circlearrowleft"] & HM \arrow[d, "\mathrm{id}", "="'] \\
& H(F_0P) \arrow[r, "H\eta_0"] & HM
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes v^1_\lambda \arrow[r, mapsto] & \text{①} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
& {[z^1_\lambda]} \arrow[r, mapsto] & 0
\end{tikzcd}$$

$\rightsquigarrow H\eta_0[z^1_\lambda] = 0$

$\rightsquigarrow \exists x^1_\lambda \in M$ s.t. $\eta_0 z^1_\lambda = dx^1_\lambda$

Define ——③

- $d : V(1) \longrightarrow F_0P$
  $v^1_\lambda \longmapsto z^1_\lambda$

$\rightsquigarrow (F_1P, d) : (R,d)$-mod

(extending $\overset{=0}{d}$ on $V(0)$)

(∵ $d(dv^1_\lambda) = d(z^1_\lambda) = 0$
$\rightsquigarrow d\circ d = 0$)

- $\eta_1 :$

$$\begin{tikzcd}
V(1) \arrow[r] & M \\
v^1_\lambda \arrow[r, mapsto] \arrow[d, mapsto, "d"] & x^1_\lambda \arrow[d, mapsto, "d"] \\
z^1_\lambda \arrow[r, mapsto, "\eta_0"] & \eta_0 z^1_\lambda = dx^1_\lambda
\end{tikzcd}$$

③

$\rightsquigarrow \eta_1 : (F_1P, d) \longrightarrow (M,d)$: chain map$/_R$

(a)(b)は明らか

(c)

$$\begin{tikzcd}
HR\otimes V(1) \arrow[r, "\delta"] \arrow[d, "\varphi_1", "\cong"'] & HR\otimes V(0) \arrow[d, "\varphi_0"', "\cong"] \\
H\left(F_1P/F_0P\right) \arrow[r, "\partial"] & H(F_0P)
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes v^1_\lambda \arrow[r, mapsto] \arrow[d, mapsto] & \delta(1\otimes v^1_\lambda) \arrow[d, mapsto] \\
& \varphi_0\circ\delta(1\otimes v^1_\lambda) \arrow[d, equal, "\text{②}"] \\
{[[1\otimes v^1_\lambda]]} \arrow[r, mapsto] & {[d(1\otimes v^1_\lambda)] = [z^1_\lambda]}
\end{tikzcd}$$

$s \geq 2$

By ind. hyp., we already have

$\eta_{s-1} : (F_{s-1}P, d) \longrightarrow (M, d)$

Take $z^s_\lambda \in F_{s-1}P$

s.t. $\begin{cases} \cdot\ d z^s_\lambda \in F_{s-2}P \\ \cdot\ [[z^s_\lambda]] = \varphi_{s-1} \circ \delta(1 \otimes v^s_\lambda) \in H\left(F_{s-1}P / F_{s-2}P\right) \end{cases}$ —— ④

Then we have a diagram:

$$\begin{tikzcd}
HR\otimes V(s) \arrow[r, "\delta"] & HR\otimes V(s-1) \arrow[r, "\partial"] \arrow[d, "\cong"', "\varphi_{s-1}"] \arrow[dr, phantom, "\circlearrowleft"] & HR\otimes V(s-2) \arrow[d, "\cong"', "\varphi_{s-2}"] \\
 & H\left(F_{s-1}/F_{s-2}\right) \arrow[r, "\partial"] & H\left(F_{s-2}/F_{s-3}\right)
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes v^s_\lambda \arrow[r, mapsto] & ④ \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
 & {[[z^s_\lambda]]} \arrow[r, mapsto] & 0
\end{tikzcd}$$

$\rightsquigarrow \partial[[z^s_\lambda]] = 0 \in H\left(F_{s-2}/F_{s-3}\right)$

Apply Lem 5.6.18 to

$\begin{cases} \cdot\ \eta = \eta_{s-1} : (F_{s-1}P, d) \longrightarrow (M, d) \\ \cdot\ k = s-1 \ (\geq 1) \\ \cdot\ z = z^s_\lambda \end{cases}$

$\rightsquigarrow \exists w^s_\lambda \in F_{s-2}P,\ \exists x^s_\lambda \in M$

s.t. $\begin{cases} \cdot\ d(z^s_\lambda - w^s_\lambda) = 0 \\ \cdot\ \eta_{s-1}(z^s_\lambda - w^s_\lambda) = d x^s_\lambda \end{cases}$ —— ⑤

Define

- $d : V(s) \longrightarrow F_{s-1}P$

  $v^s_\lambda \longmapsto z^s_\lambda - w^s_\lambda$

  $\rightsquigarrow (F_sP, d)$: (R,d)-mod

  (extending d on $F_{s-1}P$)

  ($\because\ d(d v^s_\lambda) = d(z^s_\lambda - w^s_\lambda) = 0$)

- $\eta_s : V(s) \longrightarrow M$

  $v^s_\lambda \longmapsto x^s_\lambda$

$$\begin{tikzcd}
v^s_\lambda \arrow[d, mapsto, "d"] & x^s_\lambda \arrow[d, mapsto, "d"] \\
z^s_\lambda - w^s_\lambda \arrow[r, mapsto, "\eta_{s-1}"] & \eta_{s-1}(z^s_\lambda - w^s_\lambda)
\end{tikzcd}$$

$d x^s_\lambda \ \| \ $ ⑤

$\rightsquigarrow \eta_s : (F_sP, d) \longrightarrow (M, d)$ chain map$/R$

(a)(b) は明らか

(c)
$$\begin{tikzcd}
HR\otimes V(s) \arrow[r, "\delta"] \arrow[d, "\cong"', "\varphi_s"] & HR\otimes V(s-1) \arrow[d, "\cong"', "\varphi_{s-1}"] \\
H\left(F_s/F_{s-1}\right) \arrow[r, "\partial"] & H\left(F_{s-1}/F_{s-2}\right)
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes v^s_\lambda \arrow[r, mapsto] \arrow[d, mapsto] & \delta(1\otimes v^s_\lambda) \arrow[d, mapsto] \\
{[[1\otimes v^s_\lambda]]} \arrow[r, mapsto] & {[[d(1\otimes v^s_\lambda)]] = \varphi_{s-1}\circ\delta(1\otimes v^s_\lambda)}
\end{tikzcd}$$

$\left(\because\ [d(1\otimes v^s_\lambda)] = [z^s_\lambda - \underbrace{w^s_\lambda}_{\in F_{s-2}P}] = [z^s_\lambda] \in F_{s-1}/F_{s-2} \right.$ なので、④より ok $\left.\right)$

//

以上で EM resol の存在が示された。

⑪ Eilenberg-Moore spectral sequence for Tor

Thm 5.5.6 の (R,d)-mod version が必要

Prop 5.6.20

(N, d) : right (R,d)-mod

(P, d) : left (R,d)-mod

$\{F^sP\}_s$: filtration of (P,d) as (R,d)-mod

s.t. $\forall s,\ F^sP \hookrightarrow P$: split$/R$ (ignoring diff)

$(K, d) := (N, d) \otimes_R (P, d)$

$F^sK := N \otimes_R F^sP \subset K$

Assume

- $F^{-\infty}P = P$
- $\exists s_0,\ F^{s_0}P = 0$

Then

The spectral seq for $\{F^sK\}$ satisfies:

(1) s.s. with exiting diff

(2) $E^s_1 = H\left(N \otimes_R F^sP / F^{s+1}P\right)$

(3) strongly convergent to the colimit $H(K)$

proof Thm 5.5.6 と同様 //

E_2の計算のためにLemを用意.

Lem 5.6.21 (Künneth formula)

(N,d): right (R,d)-mod

(F,d): free left (R,d)-mod

(i.e. $\exists V$: free graded $\mathbb{K}$-mod s.t. $(F,d)\cong(R,d)\otimes(V,0)$)

Then

$$HN\otimes_{HR}HF\xrightarrow{\cong}H(N\otimes_R F)\ :\ \text{isom}$$
$$[n]\otimes[f]\longmapsto[n\otimes f]$$

proof $F=R\otimes V$ は同一視する

$$(N,d)\otimes_R(F,d)=(N,d)\otimes_R((R,d)\otimes(V,0))$$
$$\cong(N,d)\otimes(V,0)$$

⇝ $H(N\otimes_R F)\cong H((N,d)\otimes(V,0))$

$$\cong HN\otimes V$$
$$\cong HN\otimes_{HR}(HR\otimes V)$$
$$\cong HN\otimes_{HR}HF$$

since V: free/$\mathbb{K}$

この isom の合成が上の map になっている //

Thm 5.6.22 (Eilenberg-Moore)

$\mathbb{K}$: comm ring, (R,d): dga/$\mathbb{K}$

(N,d): right (R,d)-mod

(M,d): left (R,d)-mod

Then

$\exists\{E_r^{st}\}$: spectral seq s.t.

- $E_2^{s,t}\cong\mathrm{Tor}^{s,t}_{HR}(HN,HM)\Rightarrow\mathrm{Tor}^{s+t}_R(N,M)$
- s.s. with exiting diff
- strongly convergent in colimit sense

proof By Cor 5.6.17, we have

$\eta:(P,d)\xrightarrow{\simeq^{q}}(M,d)$: EM resol

with $\{F_sP\}$: filtration of (P,d)

s.t. $F_sP/F_{s-1}P$: (R,d)-free

Write

$F^sP:=F_{-s}P$, $K:=N\otimes_R P$, $F^sK:=N\otimes_R F^sP$

By Prop 5.6.20,

we have s.s. $\{E_r^{st}\}$ with exiting diff, strongly convergent to colimit

Enough to show:

$$E_2^{st}\cong\mathrm{Tor}^{st}_{HR}(HN,HM)$$

Claim 1

$$\varphi^s: HN\otimes_{HR}H(F^sP/F^{s+1}P)\xrightarrow{\cong}H(F^sK/F^{s+1}K)=E_1^s$$
$$[n]\otimes[[p]]\longmapsto[[n\otimes p]]$$

(∵ $F^sK/F^{s+1}K\cong N\otimes_R\underbrace{(F^sP/F^{s+1}P)}_{(R,d)\text{-free}}$ ⇝ Lem 5.6.21)

Claim 2

$$\begin{tikzcd}
HN\otimes_{HR}H(F^sP/F^{s+1}P) \arrow[r, "\mathrm{id}\otimes\partial"] \arrow[d, "\varphi^s", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & HN\otimes_{HR}H(F^{s+1}P/F^{s+2}P) \arrow[d, "\varphi^{s+1}", "\cong"'] \\
E_1^s \arrow[r, "d_1^s"] & E_1^{s+1}
\end{tikzcd}$$

connecting hom

commutative diagram

(∵
$$\begin{tikzcd}
{[n]\otimes[[p]]} \arrow[r, mapsto, "\mathrm{id}\otimes\partial"] \arrow[dd, mapsto, "\varphi^s"] & (-1)^{|n|}[n]\otimes[[dp]] \arrow[d, mapsto, "\varphi^{s+1}"] \\
 & (-1)^{|n|}[[n\otimes dp]] \arrow[d, equal] \\
{[[n\otimes p]]} \arrow[r, mapsto, "d_1^s"] & {[[\underset{0}{\cancel{dn}}\otimes p+(-1)^{|n|}n\otimes dp]]}
\end{tikzcd}$$
$dn=0$)

Since η: EM resol, by Lem 5.6.15,

$$\cdots\to H(F^sP/F^{s+1}P)\xrightarrow{\partial}H(F^{s+1}P/F^{s+2}P)\xrightarrow{\partial}\cdots$$
$$\xrightarrow{\partial}H(F^{-1}P/F^0P)\xrightarrow{\partial}H(F^0P)\xrightarrow{H\eta}HM\to0$$

: free resol/HR

Hence, by Claim 2,

$$E_2^{st}\cong\mathrm{Tor}^{st}_{HR}(HN,HM)$$ //

Thm 5.6.1 とあわせて次を得る:

Cor 5.6.23 (Eilenberg-Moore, 66)

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

"good" pullback diagram of topological spaces

Then

$\exists\{E_r^{st}\}$: spectral sequence s.t.

- $E_2^{st} \cong \mathrm{Tor}^{st}_{H^*B}(H^*A, H^*E) \Rightarrow H^{s+t}(f^*E)$
- s.s. with exiting diff
- strongly convergent in colimit sense

Rmk 5.6.24

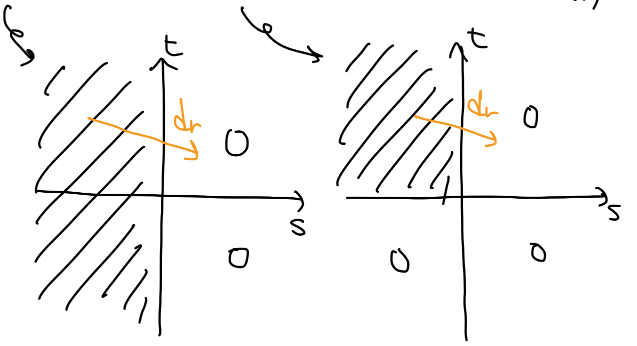
- $\mathrm{Tor}^{st}_{HR}(HN, HM) = 0$ for $s > 0$

  $(\because F^sP = F_{-s}P = 0$ for $s > 0)$

  ⇝ Tor は lower grading を使うことが多い

  しかし, このnoteでは cpx が upper grading なので, Tor もそれにあわせている

- Thm 5.6.22, Cor 5.6.23 の s.s. は下図の形,



- exiting diff なことが見てとれる
- $R = R^{\geq 0}$, $M = M^{\geq 0}$, $N = N^{\geq 0}$ だとしても bounded にはならない.

  ただし R: 1-connected なら bounded にできる

  (i.e. $R = \mathbb{K} \oplus R^{\geq 2}$)

⑩ Eilenberg-Moore spectral sequence for Ext

Thm 5.5.12 の (R,d)-mod version が必要

Prop 5.6.25

$(P,d), (N,d)$: (R,d)-mod

$\{F^sP\}_s$: filtration of (P,d) as (R,d)-mod

s.t. $\forall s$, $F^sP \hookrightarrow P$: split$/R$

(ignoring diff)

$(K,d) := \mathrm{Hom}_R((P,d),(N,d))$

$F^sK := \mathrm{Hom}_R(P/F^{-s+1}P, D) \subset K$

Assume

- $F^{-\infty}P = P$
- $\exists s_0$, $F^{s_0}P = 0$

Then

The spectral seq for $\{F^sK\}$ satisfies:

(1) s.s. with entering diff

(2) $E_1^s = H\left(\mathrm{Hom}_R\left(F^{-s}P / F^{-s+1}P, N\right)\right)$

(3) conditionally convergent to the colimit $H(K)$

~~proof~~ Thm 5.5.12 と同様 //

E_2 の計算のために Lem を用意.

Lem 5.6.26

(N,d): (R,d)-mod

(F,d): free (R,d)-mod

(i.e. $(F,d) \cong (R,d)\otimes(V,0)$)

Then

$$H(\mathrm{Hom}_R(F,N)) \xrightarrow{\cong} \mathrm{Hom}_{HR}(HF, HN)$$
$$[g] \longmapsto ([f] \mapsto [g(f)])$$

~~proof~~ $F = R\otimes V$ と同一視

$\mathrm{Hom}_R((F,d),(N,d)) \cong \mathrm{Hom}((V,0),(N,d))$

⇝ $H(\mathrm{Hom}_R(F,N)) \cong H(\mathrm{Hom}((V,0),(N,d)))$

$\cong \mathrm{Hom}(V, HN)$

$\cong \mathrm{Hom}_{HR}(HR\otimes V, HN)$

$\cong \mathrm{Hom}_{HR}(HF, HN)$

この isom の合成が上の map になっている //

Thm 5.6.27 (Eilenberg–Moore)

$\mathbb{K}$: comm ring, (R,d) : dga$/\mathbb{K}$

$(M,d), (N,d)$: (R,d)-mod

Then

$\exists \{E_r^{st}\}$: spectral seq. s.t.

- $E_2^{st} \cong \mathrm{Ext}_{HR}^{st}(HM, HN) \Rightarrow \mathrm{Ext}_R^{s+t}(M,N)$
- s.s. with entering diff
- conditionally convergent in colimit sense

proof By Prop 5.6.16, we have

$\eta : (P,d) \xrightarrow{\simeq q} (M,d)$: EM resol

with $\{F_sP\}$: filtration of (P,d)

s.t. $(F_sP/F_{s-1}P, d)$: (R,d)-free

Write

$F^sP := F_{-s}P$, $K := \mathrm{Hom}_R(P,N)$

$F^sK := \mathrm{Hom}_R(P/F^{-s+1}P, N)$

By Prop 5.6.25,

we have s.s. $\{E_r^{st}\}$ with entering diff

conditionally convergent to colimit

Enough to show:

$E_2^{st} \cong \mathrm{Ext}_{HR}^{st}(HM, HN)$

Claim 1

$$\varphi^s : \overset{E_1^s}{\overset{\shortparallel}{H\left(F^sK/F^{s+1}K\right)}} \xrightarrow{\cong} \mathrm{Hom}_{HR}\left(H\left(F^{-s}P/F^{-s+1}P\right), HN\right)$$

$$[[g]] \longmapsto H\left(g|_{F^{-s}P/F^{-s+1}P}\right)$$

$\left(g : P/F^{-s+1}P \to N\right)$

$\Big($ ∵ $F^sK/F^{s+1}K \cong \mathrm{Hom}_R\big(\underbrace{F^{-s}P/F^{-s+1}P}_{(R,d)\text{-free}}, N\big)$

⇝ Lem 5.6.26 $\Big)$

Claim 2

$$\begin{tikzcd} E_1^s \arrow[r, "d_1^s"] \arrow[d, "\varphi^s", "\cong"'] \arrow[dr, phantom, "\circlearrowleft(-1)"] & E_1^{s+1} \arrow[d, "\varphi^{s+1}", "\cong"'] \\ \mathrm{Hom}_{HR}\left(H\left(\frac{F^{-s}P}{F^{-s+1}P}\right), HN\right) \arrow[r] & \mathrm{Hom}_{HR}\left(H\left(\frac{F^{-s-1}P}{F^{-s}P}\right), HN\right) \end{tikzcd}$$

$\partial^* := \mathrm{Hom}(\partial, id)$

connecting hom

$$\begin{tikzcd} ∵\ [[g]] \arrow[r, mapsto] \arrow[dd, mapsto] & {[[d \circ g - (-1)^{|g|} g \circ d]]} \arrow[d, mapsto] \\ & \left([[p]] \mapsto [\cancel{d \circ g(p)} - (-1)^{|g|} g \circ d(p)]\right) \\ H(g|_0) \arrow[r, mapsto, "-1"] & \left([[p]] \mapsto [(-1)^{|g|} g \circ d(p)]\right) \end{tikzcd}$$

$\overset{\shortparallel}{0} \in HN$

Since η : EM resol, by Lem 5.6.15,

$$\cdots \xrightarrow{\partial} H\left(F^{-1}P/F^0P\right) \xrightarrow{\partial} H(F^0P) \xrightarrow{H\eta} HM \to 0$$

: free resol$/HR$

Hence, by Claim 2,

$E_2^{st} \cong \mathrm{Ext}_{HR}^{st}(HM, HN)$

//

Rmk 5.6.28

- grading (s? $-s$?) を"普通の" Ext と比べてみる.

  R : ring, M, N : R-mod

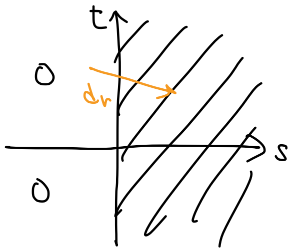
  (ungraded, without diff)

  $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$: free resol$/R$

  ⇝ $\mathrm{Ext}_R^s(M,N) = \left(\text{cohom at } \mathrm{Hom}_R(\underset{\overset{\shortparallel}{F^{-s}}}{F_s}, N)\right)$

  ⇝ 上の証明でも $F^{-s}P/F^{-s+1}P$ になっていて.

  ちゃんと consistent である.
- $\mathrm{Ext}_{HR}^{st}(HM, HN) = 0$ for $s < 0$
- s.s. は下図の形をしている:



entering diff なことが見てとれる

§5.7 Spectral sequence for Gorenstein space

Ref

[FHT88] Félix–Halperin–Thomas,
Gorenstein spaces, 1988

[FT] Félix–Thomas,
String topology on Gorenstein spaces, 2009

[Wak] Wakatsuki, Coproducts in brane topology, 2019

本題とは関係ない

see §5.6 for def of Ext over dga

Aim

収束が非自明な spectral seq を使った証明の例を与える

[Boa] の道具を使って調べたい

see [Wak, Cor3.2] for (partial) generalization

Thm 5.7.1 [FT, Thm12]

$\mathbb{K}$: field

X : 1-conn $\mathbb{K}$-Gorenstein space of dim m with $H^*(X;\mathbb{K})$: fin. type

see Def 5.7.3

$n \geq 1$

Then

$$\operatorname{Ext}^{l}_{C^*(X^n)}\left(C^*(X), C^*(X^n)\right) \overset{\text{non-canonical}}{\cong} H^{l-(n-1)m}(X)$$

where

- $C^*(-)$: the singular cochain alg $/\mathbb{K}$
- $\Delta : X \longrightarrow X^n$, $x \longmapsto (x, x, \cdots, x)$
  $\leadsto \Delta^* : C^*(X^n) \longrightarrow \underbrace{C^*(X)}_{C^*(X^n)\text{-mod}}$: dga hom

Rmk 5.7.2

$n=2$, $l=m$ のとき

$$\operatorname{Ext}^{m}_{C^*(X^2)}\left(C^*(X), C^*(X^2)\right) \cong H^0(X) \cong \mathbb{K}$$

$\Delta_!$: generator

$\leadsto$ $\Delta_!$ induces

$$H(\Delta_!) : H^l(X) \longrightarrow H^{l+m}(X^2)$$

$\overset{\text{dual}}{\leadsto}$ $H_p(X) \otimes H_q(X) \longrightarrow H_{p+q-m}(X)$

$\Delta_!$ を "intersection product" だと思うことで、

String operations がつくれる

Thm 5.7.1 の proof の概要を紹介したい

（ただし char $\mathbb{K}=0$ の case のみ）

Def 5.7.3 [FHT88]

- (R,d) : augmented dga $/\mathbb{K}$
  (i.e. $\varepsilon : (R,d) \longrightarrow \mathbb{K}$: dga hom)
  $m \in \mathbb{Z}$ に対し、
  (R,d) : Gorenstein alg of dim m
  $$\overset{\text{def}}{\Longleftrightarrow} \operatorname{Ext}^{l}_{R}(\mathbb{K}, R) \cong \begin{cases} \mathbb{K} & (l=m) \\ 0 & (l \neq m) \end{cases}$$
  where module structures are given by dga hom's
  $\varepsilon : (R,d) \longrightarrow \mathbb{K}$, $\mathrm{id} : (R,d) \longrightarrow (R,d)$
- X : based path-conn top sp に対し、
  X : $\mathbb{K}$-Gorenstein space of dim m
  $\overset{\text{def}}{\Longleftrightarrow}$ $C^*(X;\mathbb{K})$: Gorenstein alg of dim m

Example 5.7.4

- X : conn. ori. closed mfd of dim m
  $\Rightarrow$ X : Gorenstein space of dim m
- G : cpt conn. Lie group of dim n
  $\Rightarrow$ BG : Gorenstein space of dim $-n$
- X, G : as above
  $\Rightarrow$ $EG \times_G X$: Gorenstein space of dim $m-n$

Def 5.7.5

(R,d) : dga が connected

$\overset{\text{def}}{\Longleftrightarrow}$ $R = \mathbb{K} \oplus R^{\geq 1}$
(i.e. $R^{<0}=0$, $R^0=\mathbb{K}$)

$\leadsto$ canonical (unique) augmentation

Thm 5.7.6

$\mathbb{K}$: field (of $\forall$ char)

(graded) comm.

(R, d): fin. type, connected, commutative dga

(S, d): fin. type, connected Gorenstein alg of dim m

($S^0 = \mathbb{K}$ は不要)

$\varphi : (S, d) \longrightarrow (R, d)$: dga hom, augmentation-preserving

Then

$$\mathrm{Ext}^{\ell}_{R\otimes S}(R, R\otimes S) \cong H^{\ell-m}(R)$$

(where
(R, d): $(R, d)\otimes(S, d)$-mod via
$R\otimes S \xrightarrow[1\otimes\varphi]{} R\otimes R \xrightarrow[\text{multi}]{} R$: dga hom)

R: commutative

(同じ証明でもっと一般化できる [Wak, Thm 3.1])

とりあえず認めて. Thm 5.7.1 の proof

proof of Thm 5.7.1 (when char $\mathbb{K} = 0$) (sketch)

Since char $\mathbb{K} = 0$,

X: 1-conn

$\exists (R, d)$: fin. type, conn., comm., dga

s.t.

$$\begin{cases} \cdot\ (R, d) \simeq^{q} C^*(X) : \text{quasi-isom} \\ \cdot\ (R, d)^{\otimes n} \simeq^{q} C^*(X^n) : \text{—''—} \end{cases}$$

(∵ Take the minimal Sullivan model of X)

Hence

$$\mathrm{Ext}^{\ell}_{C^*(X^n)}(C^*(X), C^*(X^n))$$

$$\cong \mathrm{Ext}^{\ell}_{R^{\otimes n}}(R, R^{\otimes n})$$

$$\cong H^{\ell-(n-1)m}(R) \cong H^{\ell-(n-1)m}(X)$$

Apply Thm 5.7.6 to
$S = R^{\otimes(n-1)}$: Gorenstein alg of dim $(n-1)m$

//

idea of proof of Thm 5.7.6

- 一般に, (適切な finiteness などの仮定の下で)

$$\mathrm{Ext}_R(M, M') \otimes \mathrm{Ext}_S(N, N') \xrightarrow{\cong} \mathrm{Ext}_{R\otimes S}(M\otimes N, M'\otimes N')$$

(where
$(M\otimes N, d)$: $(R\otimes S, d)$-mod via
$(r\otimes s)\cdot(m\otimes n) := (-1)^{|s||m|}\, rm\otimes sn$)

が成立する

- $M = M' = R$, $N = \mathbb{K}$, $N' = S$ の case に上の isom を適用すると,

$$\mathrm{Ext}_R(R, R) \otimes \mathrm{Ext}_S(\mathbb{K}, S) \xrightarrow{\cong} \mathrm{Ext}_{R\otimes S}(\underset{\underset{R}{\wr\|}}{R\otimes \mathbb{K}}, R\otimes S)$$

㊟

ここで

$\cdot\ \mathrm{Ext}^i_R(R, R) \cong H^i(\mathrm{Hom}_R(R, R)) \cong H^i(R)$

$\cdot\ \mathrm{Ext}^j_S(\mathbb{K}, S) \cong \begin{cases} \mathbb{K} & (j = m) \\ 0 & (j \neq m) \end{cases}$

なので.

$$\mathrm{Ext}^{k}_{R\otimes S}(R, R\otimes S)$$

$$\xleftarrow{\cong} \mathrm{Ext}^{k-m}_R(R, R) \otimes \mathrm{Ext}^m_S(\mathbb{K}, S)$$

$$\cong H^{k-m}(R) \otimes \mathbb{K}$$

となり, Thm 5.7.6 が証明できた気がする

- しかし, 上の議論は誤りである.

実際, isom ㊟ は $R\otimes S$ mod としての isom ではない

$$\begin{cases} R\otimes S \xrightarrow{1\otimes\varepsilon} R\otimes \mathbb{K} \\ R\otimes S \xrightarrow{1\otimes\varphi} R\otimes R \xrightarrow{\text{multi}} R \end{cases}$$

spectral seq を使うことで.

上の議論を正当化する.

//

Thm 5.5.17 の (R,d)-mod ver が必要

Prop 5.7.7

$(P,d), (N,d) : (R,d)$-mod
s.t. P : free/R (ignoring diff)
← Thm 5.5.17 とは違うけど、exactness を保つには十分

$\{F^sN\}$: filtration of (N,d) as (R,d)-mod

$(K,d) := \mathrm{Hom}_{(R,d)}((P,d),(N,d))$

$F^sK := \mathrm{Hom}_R(P, F^sN) \subset K$

Assume

- $\exists s_0,\ F^{s_0}N = N$
- $F^\infty N = RF^\infty N = 0$

Then

The spectral seq for $\{F^sK\}$ satisfies:

(1) s.s. with entering diff.

(2) $E_1^s \cong H^*(\mathrm{Hom}_R(P, F^sN/F^{s+1}N))$

(3) conditionally convergent to the colimit $H^*(K)$

proof Thm 5.5.17 と同様 //

proof of Thm 5.7.6

Take

$(P,d) \xrightarrow{\simeq} (R,d)$: semifree resol/$(R\otimes S, d)$

Define

$F^s(R\otimes S) := R^{\geq s}\otimes S \subset R\otimes S$

$(R\otimes S, d)$-submod

Then we have

- $F^0(R\otimes S) = R\otimes S$
- $F^\infty(R\otimes S) = RF^\infty(R\otimes S) = 0$

(∵ **Recall** : lim, Rlim is defined degreewise
For $s>n$, $F^s(R\otimes S)^n = 0$ ← $S^{<0}=0$)

Define

$(K,d) := (\mathrm{Hom}_{R\otimes S}(P, R\otimes S), d)$

$F^sK := \mathrm{Hom}_{R\otimes S}(P, F^s(R\otimes S))$
$= \mathrm{Hom}_{R\otimes S}(P, R^{\geq s}\otimes S)$

Then, by Prop 5.7.7, we have :

$\{E_r^{st}\}$: spectral seq with entering diff

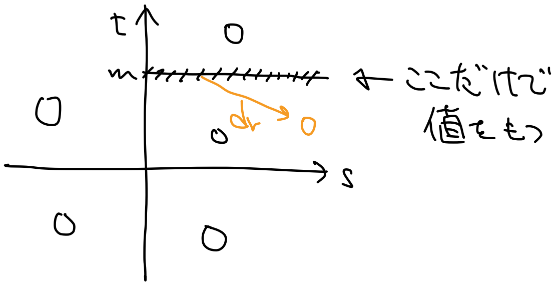
s.t.
- $E_1^{st} \cong H^{s+t}(\mathrm{Hom}_{R\otimes S}(P, R^{\geq s}/R^{\geq s+1}\otimes S))$
- conditionally conv. to the colimit $H^*(K) \cong \mathrm{Ext}_{R\otimes S}(R, R\otimes S)$

Claim

$$E_2^{st} \cong H^s(R)\otimes \mathrm{Ext}_S^t(k,S) \cong \begin{cases} H^s(R) & (t=m) \\ 0 & (t\neq m) \end{cases}$$

(・本題じゃないので証明は省略
・"idea of proof" が E_2 level で正当化された)

By Claim, E_2 is written as :



← ここだけで値をもつ

⇝ $\forall r\geq 2,\ d_r = 0$

By Cor 22.11, Prop 4.1.4, we have

$E_\infty \cong E_2,\quad RE_\infty = 0$

Hence, by Thm 4.3.1,

$\{E_r^{st}\}$: strongly convergent to the colimit $\mathrm{Ext}_{R\otimes S}(R, R\otimes S)$

⇝ $\mathrm{Ext}_{R\otimes S}(R, R\otimes S) \cong E_2$

//

Rmk 5.7.8

[FT] (や [Wak]) では、以下のようにして収束の議論を回避している：

Step 1 $N \in \mathbb{N}$ を fix し、

$$K^{(N)} := \mathrm{Hom}_{R\otimes S}\left(P, R/R^{\geq N} \otimes S\right)$$

$$F^s K^{(N)} := \begin{cases} \mathrm{Hom}_{R\otimes S}\left(P, R^{\geq s}/R^{\geq N} \otimes S\right) & (s \leq N) \\ 0 & (s > N) \end{cases}$$

と def.

$\rightsquigarrow$ $\{F^s K^{(N)}\}_s$: finite filtration なので、

spectral seq は明らかに strongly conv.

$$\rightsquigarrow H^l(K^{(N)}) \cong H^{l-m}\left(R/R^{\geq N}\right)$$

Step 2 $\varprojlim_N K^{(N)} = K$, $R\varprojlim_N K^{(N)} = 0$

$$R\varprojlim_N H(K^{(N)}) = 0$$

$$\longrightarrow H^l(K) \cong \varprojlim_N H^l(K^{(N)}) = H^{l-m}(R)$$

- conditional convergence を使えば、このような小細工は不要になる
  （けど、「小細工」の方が前提知識は少ないので読者にやさしい）
- $K^{(N)} = K/F^N K$ なので、§5.4 の状況になっている
  とくに、Thm 5.4.13 (3) より、次の2つが関係している：
  - $\{E_r\}$ (for $\{F^s K\}_s$) : strongly convergent
  - $R\varprojlim_N H(K^{(N)}) = 0$ ← Thm 5.4.13 の記号では A^∞

§5.8 Atiyah-Hirzebruch spectral sequence

[Boa, §12]

(後半の内容は §5.4 を参照)

homology version:

Thm 5.8.1 (Atiyah-Hirzebruch)

$h_*(-)$: (unreduced) homology theory ← see Def 5.8.4

X: CW cpx

Then

$\exists \{E^r_{st}\}$: spectral seq s.t.

- $E^2_{st} \cong H_s(X; h_t(pt)) \Rightarrow h_{s+t}(X)$
- s.s. with exiting diff.
- strongly convergent in colimit sense

↖ $H_s(-; \underset{\text{ab. grp.}}{h_t(pt)})$: singular homology with coeff. in $h_t(pt)$

cohomology version:

Thm 5.8.2 (Atiyah-Hirzebruch)

$h^*(-)$: (unreduced) cohomology theory ← see Def 5.8.32

X: CW cpx

Then

$\exists \{E_r^{st}\}$: spectral seq s.t.

- $E_2^{st} \cong H^s(X; h^t(pt)) \Rightarrow h^{s+t}(X)$
- s.s. with entering diff.
- conditionally convergent in colimit sense

↖ $H^s(-; \underset{\text{ab. grp.}}{h^t(pt)})$: singular cohomology with coeff. in $h^t(pt)$

Rmk 5.8.3

"X: finite CW cpx" を仮定する文献もあるが、
[Boa] の理論を使えばその仮定は不要

◎ Homology theory

Def 5.8.4

(h_*, ∂): (generalized) homology theory

$\overset{\text{def}}{\Longleftrightarrow}$

- $h_* = \{h_n\}_{n\in\mathbb{Z}}$

  h_n: CWpair $\longrightarrow$ Ab : functor

  Obj (X,A) $A\subset X$: subcpx
  Mor conti. map
  abelian groups

  Write $h_n(X) := h_n(X, \emptyset)$
- $\partial = \{\partial_n\}_{n\in\mathbb{Z}}$

  $\partial_n : h_n(X,A) \longrightarrow h_{n-1}(A)$

  natural transformation on (X,A)

satisfying the following axioms:

homotopy invariant

$f, g : (X,A) \longrightarrow (Y,B)$

$f \simeq g$: homotopic

$\Rightarrow h_n(f) = h_n(g)$

exact sequence

$\cdots \to h_n(A) \to h_n(X) \to h_n(X,A) \xrightarrow{\partial} h_{n-1}(A) \to \cdots$: exact

excision

$A, B \subset X$: subcpx に対し、

$h_n(A, A\cap B) \xrightarrow{\cong} h_n(A\cup B, B)$

additive

$\{(X_\lambda, A_\lambda)\}_{\lambda\in\Lambda}$: family of CW pair

$\bigoplus_\lambda h_n(X_\lambda, A_\lambda) \xrightarrow{\cong} h_n(\amalg (X_\lambda, A_\lambda))$

Example 5.8.5

- $h_n(-) := H_n(-; M)$
- stable homotopy group

Lem 5.8.6

$h_*(X, A) \xrightarrow{\cong} h_*(X/A, *)$

the canonical base pt of X/A

~~proof~~ 略 //

Lem 5.8.7

$\{X_\lambda\}_{\lambda\in\Lambda}$: family of based sp

Then

$$\bigoplus_\lambda h_n(X_\lambda,*) \xrightarrow{\cong} h_n\Big(\bigvee_\lambda X_\lambda,*\Big)$$

proof 略 //

Lem 5.8.8

$X \supset A \supset B$ に対し、

$$\cdots \to h_n(A,B) \to h_n(X,B) \to h_n(X,A) \xrightarrow{\partial} h_{n-1}(A,B) \to \cdots \quad \text{: exact}$$

proof Define ∂ by

$$h_n(X,A) \xrightarrow{\partial} h_{n-1}(A) \xrightarrow{i_*} h_{n-1}(A,B)$$

$$\big(i : A = (A,\emptyset) \hookrightarrow (A,B)\big)$$

//

Def 5.8.9

X : CW cpx に対し、 ← ここだけの記号

$$\overline{h}_n(X) := \mathrm{Ker}\big(\varepsilon_* : h_n(X) \to h_n(*)\big)$$

(where $\varepsilon : X \to *$: the unique map
$*$ ← terminal obj)

Lem 5.8.10

$(X,*)$: based CW cpx

Then

$h_n(X) \to h_n(X,*)$ induces

$$\overline{h}_n(X) \xrightarrow{\cong} h_n(X,*) \text{ : isom}$$

proof

$$\begin{tikzcd}
0 \arrow[r] & h_n(*) \arrow[r] \arrow[d] & h_n(X) \arrow[r] \arrow[d] & h_n(X,*) \arrow[r] \arrow[d] & 0 & \text{exact} \\
0 \arrow[r] & h_n(*) \arrow[r] & h_n(*) \arrow[r] & 0 & \text{exact} &
\end{tikzcd}$$

(∵ $* \hookrightarrow X$ has a section $\varepsilon : X \to *$
⇝ exactness of the upper row)

⇝ snake lemma //

Rmk 5.8.11

$\tilde{h}_n(X) := h_n(X,*)$: reduced homology

- 今回は必ず右辺で書く
- based にしたくない場面では、
  $h_n(X,*)$ の代わりに $\overline{h}_n(X)$ を使う

(とくに $X = S^0 = \{+1,-1\}$ のとき、
± 1 は対称的に扱いたい)

CW cpx を扱うための準備として、
sphere を調べておく:

Lem 5.8.12

$\forall s \geq 0$, $\forall n$,

$$\partial : h_n(D^s, S^{s-1}) \to h_{n-1}(S^{s-1})$$

restricts to

$$\partial : h_n(D^s, S^{s-1}) \xrightarrow{\cong} \overline{h}_{n-1}(S^{s-1})$$

isom

proof

$$\begin{tikzcd}
 & & & h_n(*) & \\
\cdots \arrow[r] & h_n(S^{s-1}) \arrow[r] \arrow[urr, "\varepsilon_*"] \arrow[rr, phantom, bend left, "\circlearrowleft"] & h_n(D^s) \arrow[r] \arrow[ur, "\varepsilon_*"', "\cong"] & h_n(D^s, S^{s-1}) &
\end{tikzcd}$$

$$\xrightarrow{\partial} h_{n-1}(S^{s-1}) \to \cdots \quad \text{: exact}$$

$$\varepsilon_* : h_{n-1}(S^{s-1}) \twoheadrightarrow h_{n-1}(*) \text{ : surj}$$

(∵ ∃ section $* \to S^{s-1}$)

Hence

$$0 \to h_n(D^s, S^{s-1}) \xrightarrow{\partial} h_{n-1}(S^{s-1}) \xrightarrow{\varepsilon_*} h_{n-1}(*) \to 0$$

: exact

⇝ ∂ : inj and

$$\mathrm{Im}\,\partial = \mathrm{Ker}\,\varepsilon_* = \overline{h}_{n-1}(S^{s-1})$$

//

Lem 5.8.13

$\exists \{\Sigma^h_{st}\}_{s\geq 0, t\in\mathbb{Z}}$: family of isom's

$$\Sigma^h_{st} : h_t(*) \xrightarrow{\cong} h_{s+t}(D^s, S^{s-1})$$

s.t. $\forall s \geq 1$, $\forall t \in \mathbb{Z}$,

the following diagram commutes:

$$\begin{tikzcd}
h_t(*) \arrow[r, "\Sigma^h_{st}", "\cong"'] \arrow[d, "\cong"', "\Sigma^h_{s-1,t}"] \arrow[dr, phantom, "\circlearrowleft"] & h_{s+t}(D^s, S^{s-1}) \arrow[d, "\cong"', "\partial"] \\
 & \tilde{h}_{s+t-1}(S^{s-1}) \arrow[d, "\cong"', "\leftarrow \text{Lem 5.8.10}"] \\
h_{s+t-1}(D^{s-1}, S^{s-2}) \arrow[r, "\cong"] & h_{s+t-1}(S^{s-1}, *)
\end{tikzcd}$$

← induced by quotient map

~~proof~~ Construct Σ^h_{st} by induction on s

$\underline{s=0}$ $\Sigma^h_{0t} = \mathrm{id} : h_t(*) \to h_t(*, \phi)$

$\underline{s\geq 1}$ Define Σ^h_{st} by the above diagram //

Def 5.8.14

$s \geq 0$

$f : S^s \to S^s$: continuous map

Define

$\deg f \in \mathbb{Z}$: mapping degree

as follows:

Consider the map

$f_* : \tilde{H}_s(S^s) \to \tilde{H}_s(S^s)$

with coeff. in $\mathbb{Z}$

Since $\tilde{H}_s(S^s) \cong \mathbb{Z}$,

$\exists ! (\deg f) \in \mathbb{Z}$ s.t. $f_*(\omega) = (\deg f)\cdot\omega$

Rmk 5.8.15

- $s > 0$ ならいつものやつと同じ
- $s = 0$ のときも、これで def する

(based でない map も考えるために、$H_s(S^s, *)$ ではなく $\tilde{H}_s(S^s)$ を使った)

Lem 5.8.16

Write $S^0 = \{+1, -1\}$

(1) $\mathrm{Map}(S^0, S^0) \xrightarrow{\cong} \{\pm 1\} \times \{\pm 1\}$: bij

$f \longmapsto (f(1), f(-1))$

(2) Under the above bij,

we have the table of $\deg f$:

$f(1)$ \ $f(-1)$	1	-1
1	0	-1 ← -1倍写像
-1	1 ← id	0

~~proof~~ 明らか //

Fact 5.8.17

Assume $s \geq 1$

Then

$$[S^s, S^s] \xrightarrow{\cong} \mathbb{Z} \text{ : bij}$$
$$f \longmapsto \deg f$$

← $\{\text{maps } S^s \to S^s\}/\text{homotopy}$

← map は based, unbased どちらでも ok

各 $n \in \mathbb{Z}$ に対し、$\deg f = n$ となる f を具体的に与えたい

Lem 5.8.18

$s \geq 1$

$f, g : S^s \to S^s$

Define

$$\begin{cases} f+g : S^s \xrightarrow{\Delta} S^s \vee S^s \xrightarrow{f\vee g} S^s \\ -f : S^s \xrightarrow{r} S^s \xrightarrow{f} S^s \end{cases}$$

(where Δ : pinch map

r : orientation reversing map)

Then $\forall n \in \mathbb{Z}$

$$\begin{cases} (f+g)_* = f_* + g_* : \tilde{h}_n(S^s) \to \tilde{h}_n(S^s) \\ (-f)_* = -f_* : \quad —〃— \end{cases}$$

~~proof~~ $\Delta_* : \tilde{h}_n(S^s) \to \tilde{h}_n(S^s \vee S^s) \xrightarrow{\cong} \tilde{h}_n(S^s) \oplus \tilde{h}_n(S^s)$

$\omega \longmapsto (\omega, \omega)$

$\rightsquigarrow (f+g)_* = f_* + g_*$

また、$\mathrm{id} + r \simeq \mathrm{const}$ より、$\mathrm{id}_* + r_* = (\mathrm{id} + r)_* = 0$ //

Prop 5.8.19

$s \geq 0$, $f : S^s \longrightarrow S^s$: conti

Then

$\forall n \in \mathbb{Z}$, $f_* = (\deg f)\cdot : \tilde{h}_n(S^s) \to \tilde{h}_n(S^s)$

proof

$s = 0$ Lem 5.8.17と同様に、4通り計算する。

$s \geq 1$

$k \in \mathbb{Z}$ に対し、

$$f_k := \begin{cases} \overbrace{(\cdots(id + id) + \cdots) + id}^{k個} & (k > 0) \\ const & (k = 0) \\ -f_{|k|} & (k < 0) \end{cases}$$

と定めると、Lem 5.8.19 と $id_* = 1\cdot$ より、

$$(f_k)_* = k\cdot : \tilde{h}_n(S^s) \to \tilde{h}_n(S^s)$$

$\rightsquigarrow \deg(f_k) = k$

($\because$ $h_* := H_*(-;\mathbb{Z})$ の case を考えればよい)

よって h_* の homotopy invariance と Fact 5.8.16 から主張が従う。 //

最後に h_* と colim の可換性について：

Fact 5.8.20 (Milnor, [Boa, Thm 4.2])

X: CW cpx

$\cdots \subset F_sX \subset F_{s+1}X \subset \cdots \subset X$

filtration by subcpx's s.t. $\bigcup_s F_sX = X$

Then

(a) $\operatorname{colim}_s h_n(F_sX) \xrightarrow{\cong} h_n(X)$

(b) $\operatorname{colim}_s h_n(X, F_sX) = 0$

c.f. Thm 3.3.2 (1)

idea of proof

$T :=$ (mapping telescope of $\{F_sX\}_s$)

$\rightsquigarrow T \simeq \operatorname{hocolim}_s F_sX \simeq X$

$$\bigvee_s X_s \longrightarrow T \longrightarrow \bigvee_s \Sigma X_s \xrightarrow{1-i} \bigvee_s \Sigma X_s$$

→ colim

cofiber seq //

Review on cellular homology

X: CW cpx

Notation

普通は $X^{(s)}$ と書くかも

- $X_s := \bigcup$ (cells of dim $\leq s$)
  s-skeleton of X
  (• $X_{-1} := \phi$
  • increasing filtration of X)
- $\{e^s_\alpha\}_{\alpha \in \Lambda^s}$: the set of s-cells
  index set
- For $\alpha \in \Lambda^s$,
  $\varphi^s_\alpha : (D^s, S^{s-1}) \longrightarrow (X_s, X_{s-1})$
  $\quad\quad\quad\ \ (D^s_\alpha, S^{s-1}_\alpha)$ ($\parallel$) attaching map for e^s_α
  ← 正確には $\varphi^s_\alpha|_{S^{s-1}_\alpha}$ のこと?

Def 5.8.21

M: ab. grp.

$(C^{cell}_*(X;M), \partial)$: chain cpx を次で def:

- $C^{cell}_s(X;M) := H_s(X_s, X_{s-1}; M)$
- $\partial : C^{cell}_s(X;M) \longrightarrow C^{cell}_{s-1}(X;M)$
  connecting for (X_s, X_{s-1}, X_{s-2})

Rmk 5.8.22

see Cor 5.8.30

$(C^{cell}_*(X;M), \partial)$ の記述は、一般の homology theory とあわせてやる

- $C^{cell}_s(X;M) \cong H_s(X_s, X_{s-1}; \mathbb{Z}) \otimes_{\mathbb{Z}} M$
  $\cong \mathbb{Z}^{\oplus \Lambda^s} \otimes_{\mathbb{Z}} M$
- ∂ is determined by the mapping degree of some maps $S^{s-1} \longrightarrow S^{s-1}$

Lem 5.8.23

$$\bigvee_{\alpha\in\Lambda^s}\varphi^s_\alpha : \bigvee_{\alpha\in\Lambda^s} S^s_\alpha = \bigvee_{\alpha\in\Lambda^s} D^s_\alpha / S^{s-1}_\alpha \xrightarrow[\text{homeo}]{\cong} X_s / X_{s-1}$$

proof 略//

(※ $s=0$ でも成立する
($\because S^0 = \{+1,-1\}$, $Y/\phi = Y \amalg \{*\}$))

Prop 5.8.24

$\forall s$, $H_s(C^{cell}_*(X;M), \partial) \cong H_s(X;M)$

proof 係数の M は省略する

X: filtered space by $\{X_s\}_s$ これは bounded ではない

$\rightsquigarrow$ $\underset{\text{※ singular chain cpx}}{C_*(X)}$: filtered cpx by $\{C_*(X_s)\}_s$

$\rightsquigarrow$ $(A_s := H_*(X_s),\ E_s := H_*(X_s, X_{s-1}))$
unrolled exact couple

$\rightsquigarrow$ $\{E^r_{st}\}$: spectral seq

Then we have

- $\forall s<0$, $A_s = 0$ ($\because X_s = \phi$)
- $E_{st} = H_{s+t}(X_s, X_{s-1})$

$$= \begin{cases} C^{cell}_s(X) & (t=0) \\ 0 & (t\neq 0) \end{cases}$$

※($\because$ Fact 5.8.6)

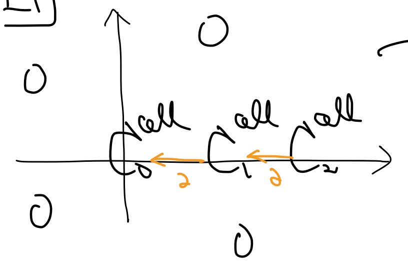
Hence 実際には "bounded exact couple"

$\{E^r_{st}\}$: spectral seq with exiting diff

strongly convergent to

$\operatorname{colim}_s A_s \cong H_*(X)$

E_1



$\rightsquigarrow$ $E^2 \cong H_*(C^{cell}_*(X))$
$\cong$
E^∞

By strong convergence,

this proves the Prop //

Rmk 5.8.25

Prop 5.8.7 の証明の鍵は,

$E_{st} = 0$ for $t \neq 0$

($\rightsquigarrow d^r = 0$ for $r \geq 2$)

であったが、これは Fact 5.8.6 に加えて

$H_{s+t}(S^s) = 0$ for $t \neq 0$

が成り立つことを用いている.

$\rightsquigarrow$ 一般の homology theory だと
これが不成立なので $d^r \neq 0$ となり,
Thm 5.8.1 の形になる

⑩ Atiyah-Hirzebruch s.s. for homology

X: CW cpx

Def 5.8.26

$s \geq 1$, $\alpha \in \Lambda^s$, $\beta \in \Lambda^{s-1}$ に対し,

- $\psi^s_{\alpha\beta} : S^{s-1}_\alpha \longrightarrow S^{s-1}_\beta$ を次の合成で def:

$$\begin{tikzcd} S^{s-1}_\alpha \arrow[r, "\varphi_\alpha"] & X_{s-1} \arrow[r, two heads] & X_{s-1}/X_{s-2} & \bigvee_{\beta'\in\Lambda^{s-1}} S^{s-1}_{\beta'} \arrow[l, "\bigvee \varphi^{s-1}_{\beta'}"', "\cong"] \arrow[r, two heads] & S^{s-1}_\beta \end{tikzcd}$$

- $n^s_{\alpha\beta} := \deg(\psi^s_{\alpha\beta}) \in \mathbb{Z}$

Lem 5.8.27

$\forall \alpha \in \Lambda^s$ に対し, 次が成立:

(1) $\#\{\beta \in \Lambda^{s-1} \mid \psi^s_{\alpha\beta} \neq (\text{const at } *)\} < \infty$

(2) $\#\{\beta \in \Lambda^{s-1} \mid n^s_{\alpha\beta} \neq 0\} < \infty$

proof

(1) S^{s-1}_α: cpt と S^{s-1}_β: T_1 から従う.
(i.e. $\forall x \in S^{s-1}_\beta$, $\{x\}$: closed)

(2) $\deg(\text{const}) = 0$ なので (1) より OK //

Lem 5.8.28

M: ab. grp

Define

$$M_\alpha = M_\beta := M \quad (\alpha \in \Delta^s, \beta \in \Delta^{s-1})$$

Then

the following maps are well-def'd:

(1) $(n^s_{\alpha\beta})_{\alpha\beta} : \bigoplus_{\alpha \in \Delta^s} M_\alpha \longrightarrow \bigoplus_{\beta \in \Delta^{s-1}} M_\beta$

$$\{x_\alpha\}_\alpha \longmapsto \left\{\sum_\alpha n^s_{\alpha\beta} x_\alpha\right\}_\beta$$

↑ 行列のつもり

(2) $(n^s_{\alpha\beta})_{\beta\alpha} : \prod_{\beta \in \Delta^{s-1}} M_\beta \longrightarrow \prod_{\alpha \in \Delta^s} M_\alpha$

$$\{y_\beta\} \longmapsto \left\{\sum_\beta n^s_{\alpha\beta} y_\beta\right\}_\alpha$$

↑ 上のやつの転置

proof Lem 5.8.27 (2) //

Prop 5.8.29

For $s \geq 1$,

the following diagram commutes:

$$\begin{tikzcd}
h_{s+t}(X_s, X_{s-1}) \arrow[r, "\partial"] & h_{s+t-1}(X_{s-1}, X_{s-2}) \\
\bigoplus_\alpha h_{s+t}(D^s_\alpha, S^{s-1}_\alpha) \arrow[u, "\bigoplus_\alpha \varphi^s_{\alpha *}", "\cong"'] & \bigoplus_\beta h_{s+t-1}(D^{s-1}_\beta, S^{s-2}_\beta) \arrow[u, "\cong", "\bigoplus_\beta \varphi^{s-1}_{\beta *}"'] \\
\bigoplus_\alpha h_t(*) \arrow[u, "\bigoplus_\alpha \Sigma^h_{st}", "\cong"'] \arrow[r, "(n^s_{\alpha\beta})_{\alpha\beta}"'] \arrow[ur, phantom, "\circlearrowright"] & \bigoplus_\beta h_t(*) \arrow[u, "\bigoplus_\beta \Sigma^h_{s-1,t}"']
\end{tikzcd}$$

proof of Prop 5.8.29 By Lem 5.8.13, we have

$$\begin{tikzcd}
h_t(*) \arrow[rr, "\Sigma^h_{st}", "\cong"'] \arrow[d, "\Sigma^h_{s-1,t}", "\cong"'] \arrow[drr, "\tilde{\Sigma}^h_{st}", "\cong"'] & & h_{s+t}(D^s, S^{s-1}) \arrow[d, "\partial", "\cong"'] \\
h_{s+t-1}(D^{s-1}, S^{s-2}) \arrow[r, "\cong"] & h_{s+t-1}(S^{s-1}, *) & \tilde{h}_{s+t-1}(S^{s-1}) \arrow[l, "\cong"']
\end{tikzcd}$$

define

↻ ↻

$$\begin{tikzcd}
h_{s+t}(X_s, X_{s-1}) \arrow[r, "\partial"] & h_{s+t-1}(X_{s-1}, X_{s-2}) \arrow[rr, "\mathrm{id}", "="'] \arrow[dr, "\cong"] & & h_{s+t-1}(X_{s-1}, X_{s-2}) \arrow[dl, "\cong"'] \\
& & h_{s+t-1}(X_{s-1}/X_{s-2}, *) & \\
& & h_{s+t-1}(\bigvee_\beta S^{s-1}_\beta, *) \arrow[u, "\cong", "(\bigvee_\beta \varphi^{s-1}_\beta)_*"'] & \\
& & \bigoplus_\beta h_{s+t-1}(S^{s-1}_\beta, *) \arrow[u, "\cong"] & \\
\bigoplus_\alpha h_{s+t}(D^s_\alpha, S^{s-1}_\alpha) \arrow[uuuu, "\bigoplus_\alpha \varphi^s_{\alpha *}", "\cong"'] \arrow[r, "\partial", "\cong"'] & \bigoplus_\alpha \tilde{h}_{s+t-1}(S^{s-1}_\alpha) \arrow[uuuu, "\bigoplus_\alpha \varphi^s_{\alpha *}"] \arrow[r, bend left, "((\psi^s_{\alpha\beta})_*)_{\alpha\beta}"] \arrow[r, bend right, "(n^s_{\alpha\beta})_{\alpha\beta}"'] & \bigoplus_\beta \tilde{h}_{s+t-1}(S^{s-1}_\beta) \arrow[u, "\cong"] & \bigoplus_\beta h_{s+t-1}(D^{s-1}_\beta, S^{s-2}_\beta) \arrow[uuuu, "\cong", "\bigoplus_\beta \varphi^{s-1}_{\beta *}"'] \arrow[ul, "\cong"'] \\
\bigoplus_\alpha h_t(*) \arrow[u, "\bigoplus_\alpha \Sigma^h_{st}", "\cong"'] \arrow[ur, "\cong", "\bigoplus_\alpha \tilde{\Sigma}^h_{st}"'] \arrow[rrr, "(n^s_{\alpha\beta})_{\alpha\beta}"'] & & & \bigoplus_\beta h_t(*) \arrow[u, "\cong", "\bigoplus_\beta \Sigma^h_{s-1,t}"'] \arrow[ul, "\bigoplus_\alpha \tilde{\Sigma}^h_{st}", "\cong"']
\end{tikzcd}$$

↻ 明らか

↻ naturality of ∂

↻ def of $\psi^s_{\alpha\beta}$

↻ 明らか

↻ def of $\tilde{\Sigma}^h_{st}$

↻ Prop 5.8.19

↻ def of $\tilde{\Sigma}^h_{st}$

↻ 明らか

//

Cor 5.8.30

M: ab. grp.

Then $^{\exists}\{\Phi^M_s\}_{s\geq 1}$: family of isom's s.t.

$$\begin{tikzcd}
C^{\mathrm{cell}}_s(X;M) \arrow[r, "\partial"] & C^{\mathrm{cell}}_{s-1}(X;M) \\
\bigoplus_\alpha M \arrow[u, "\cong"', "\Phi^M_s"] \arrow[r, "(n^s_{\alpha\beta})_{\alpha\beta}"'] \arrow[ur, phantom, "\circlearrowleft"] & \bigoplus_\beta M \arrow[u, "\cong"', "\Phi^M_{s-1}"]
\end{tikzcd}$$

proof

Apply Prop 5.8.29 to the case

$h_*(-) = H_*(-;M)$, $t = 0$

//

Cor 5.8.31

$\exists\{\Psi^h_{st}\}_{s\geq 1, t\in\mathbb{Z}}$: family of isom's s.t.

$$\begin{tikzcd}
h_{s+t}(X_s, X_{s-1}) \arrow[r, "\partial"] & h_{s+t-1}(X_{s-1}, X_{s-2}) \\
C^{\mathrm{cell}}_s(X;h_t(*)) \arrow[u, "\cong", "\Psi^h_{st}"'] \arrow[r, "\partial"] \arrow[ur, phantom, "\circlearrowleft"] & C^{\mathrm{cell}}_{s-1}(X;h_t(*)) \arrow[u, "\cong", "\Psi^h_{s-1,t}"']
\end{tikzcd}$$

proof

Prop 5.8.29 + Prop 5.8.30 (for $M = h_t(*)$)

//

以上で Thm 5.8.1 の証明の準備が整った.

proof of Thm 5.8.1

Define

$$\begin{cases} A_{st} := h_{s+t}(X_s) \\ E_{st} := h_{s+t}(X_s, X_{s-1}) \end{cases}$$

$\rightsquigarrow$

$$\begin{tikzcd}
\cdots \arrow[r] & {} & A_{s+1} \arrow[r] \arrow[dl] & {} & A_s \arrow[r] \arrow[dl] & {} & \cdots \\
& {} & & E_s \arrow[ul, "\partial"] & & {} \arrow[ul] &
\end{tikzcd}$$

unrolled exact couple

$\rightsquigarrow$ $\{E^r_{st}\}$: spectral seq.

Then we have:

- $\{E^r_{st}\}$: spectral seq with exiting diff strongly convergent to $\operatorname{colim}_s A_s$

  (☺ $^\forall s<0$, $A_s = h_t(\phi) = 0$)

- $\operatorname{colim}_s A_s = \operatorname{colim}_s h_*(X_s) \overset{\uparrow}{\cong} h_*(X)$

  (☺ Fact 5.8.20 (ii))

- $E^2_{st} \cong H_s(X; h_t(*))$

  (☺ $d^1_{st} = \partial: h_{s+t}(X_s, X_{s-1}) \to h_{s+t-1}(X_{s-1}, X_{s-2})$

  $\rightsquigarrow$ Cor 5.8.31)

//

● Cohomology theory

基本的に homology theory の "dual" を書くだけなので, 大半を省略する.

Def 5.8.32

- (h^*, δ): (generalized) cohomology theory

$\underset{\mathrm{def}}{\Longleftrightarrow}$

- h^n: $\mathrm{CWpair}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$: functor
- δ^n: $h^n(A) \longrightarrow h^{n+1}(X,A)$ nat. trans.

s.t.

homotopy invariant

exact sequence

excision

additive

$$h^n\left(\coprod_\lambda (X_\lambda, A_\lambda)\right) \xrightarrow{\cong} \prod_\lambda h^n(X_\lambda, A_\lambda)$$

- $\tilde{h}^n(X) := \operatorname{Coker}(\varepsilon^*: h^n(*) \longrightarrow h^n(X))$

Fact 5.8.33 (Milnor, [Boa, Thm 4.3])

X: CW cpx

$\cdots \subset F_sX \subset F_{s+1}X \subset \cdots \subset X$

filtration by subcpx's s.t. $\bigcup_s F_sX = X$

Then

(1) $0 \to R\lim_s h^{n-1}(F_sX) \longrightarrow h^n(X) \longrightarrow \lim_s h^n(F_sX) \to 0$: exact

(2) $\lim_s h^n(X, F_sX) = R\lim_s h^n(X, F_sX) = 0$

(c.f. Thm 3.3.2 (2))

◎ Atiyah-Hirzebruch s.s. for cohomology

基本的には homology と同じ

ただし、$\prod_\beta M_\beta \longrightarrow \prod_\alpha M_\alpha$ の取扱いに注意が必要

Def 5.8.34

$\{M_\alpha\}_{\alpha\in A}$, $\{N_\beta\}_{\beta\in B}$: family of ab. grp

$f : \prod_\beta N_\beta \longrightarrow \prod_\alpha M_\alpha$

について、 ← ここだけの用語

f : finitely determined

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \alpha \in A$, $\exists B_\alpha \subset B$: finite subset

$\exists f_\alpha : \prod_{\beta\in B_\alpha} N_\beta \longrightarrow M_\alpha$

s.t.

$$\begin{tikzcd} \prod_\beta N_\beta \arrow[r, "f"] \arrow[d, "pr"] & \prod_{\alpha'} M_{\alpha'} \arrow[r, "pr"] & M_\alpha \\ \prod_{\beta\in B_\alpha} N_\beta \arrow[urr, dashed, "f_\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & & \end{tikzcd}$$

Lem 5.8.35

$f : \prod_\beta N_\beta \longrightarrow \prod_\alpha N_\alpha$: fin. determined

Define

$$f_{\beta\alpha} : N_\beta \hookrightarrow \prod_{\beta'} N_{\beta'} \xrightarrow{f} \prod_{\alpha'} M_{\alpha'} \twoheadrightarrow M_\alpha$$

Then

$\forall \{n_\beta\}_\beta \in \prod_\beta N_\beta$,

$$f(\{n_\beta\}) = \Big\{\underbrace{\sum_\beta f_{\beta\alpha}(n_\beta)}\Big\}_\alpha$$

← well-def'd

proof 明らか //

Prop 5.8.36

For $s \geq 1$,

$$\begin{tikzcd} h^{s+t-1}(X_{s-1}, X_{s-2}) \arrow[r, "\delta"] & h^{s+t}(X_s, X_{s-1}) \\ \prod_{\beta\in\Lambda^{s-1}} h^t(*) \arrow[u, "\cong"] \arrow[r, "(n^s_{\alpha\beta})_{\beta\alpha}"'] \arrow[ur, phantom, "\circlearrowleft"] & \prod_{\alpha\in\Lambda^s} h^t(*) \arrow[u, "\cong"'] \end{tikzcd}$$

sketch of proof Prop 5.8.29 と同様の図式を書く。可換性を示す際に、Lem 5.8.27 (1) より

$\prod_\beta \widetilde{h}^{s+t-1}(S^{s-1}_\beta) \longrightarrow \prod_\alpha \widetilde{h}^{s+t-1}(S^{s-1}_\alpha)$: finitely determined

なので、Lem 5.8.35 より成分ごとに計算すればよい。 //

proof of Thm 5.8.3

Define

$$\begin{cases} A^{st} := h^{s+t}(X, X_{s-1}) \\ E^{st} := h^{s+t}(X_s, X_{s-1}) \end{cases}$$

$$\begin{tikzcd} \cdots \arrow[r] & A^{s+1} \arrow[r] \arrow[dl] & & A^s \arrow[r] \arrow[dl] & \cdots \\ {} & & E^s \arrow[ul] & & {} \arrow[ul] \end{tikzcd}$$

⇝ : unrolled exact couple

⇝ $\{E^{st}_r\}$: spectral seq.

Then we have:

- $\{E^{st}_r\}$: spectral seq with entering diff conditionally convergent to $\operatorname{colim}_s A^s$

  (∵ entering diff

  $\forall s < 0$, $A^s = h^*(X, \emptyset) = h^*(X)$

  cond. conv.

  By Fact 5.8.33 (2),

  $\lim_s A^s = R\lim_s A^s = 0$)

- $\operatorname{colim}_s A^s = h^*(X)$

  (∵ $\forall s < 0$, $A^s = h^*(X)$)

- $E^{st}_2 \cong H^s(X; h^t(*))$

  (∵ Prop 5.8.36)

//