Homotopy associative algebras

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§1. Introduction

Notations

- coefficients : a field $\mathbb{K}$
- $M = \{M_n\}_{n\in\mathbb{Z}}$: graded ($\mathbb{K}$-) module
  For $x \in M_n$, we denote $|x| := n$ degree
- $f = \{f_n : M_n \to N_{n+k}\}_{n\in\mathbb{Z}} : M \to N$
  : linear map of deg k

Def 1.1 (dg algebra)

(A, μ, d) : dg algebra (differential graded algebra)

$\overset{\text{def}}{\iff}$

- (A, μ) : graded algebra
  (i.e. $\mu : A\otimes A \to A$, $x\otimes y \mapsto xy$: linear map of deg 0 satisfying associativity)
- $d : A \to A$: linear map of deg (-1)
  which is
  - differential (i.e. $d^2 = 0$)
  - derivation (Leibniz rule)
    (i.e. $d\mu = \mu(d\otimes 1_A + 1_A \otimes d)$)

Rmk 1.2

$d\mu(x\otimes y) = d(xy)$

$\mu(d\otimes 1_A + 1_A\otimes d)(x\otimes y) = \mu(dx\otimes y + (-1)^{|x|}x\otimes dy)$
$= dx\cdot y + (-1)^{|x|}x\cdot dy$

Ex 1.3

- $A^*_{dR}(M)$: de Rham algebra for M : mfd
- $C^*_{sing}(X)$: singular cochain algebra for X : top. space

$(A^n = A_{-n})$

Def 1.4 (A_∞-algebra)

(A, m) : A_∞-algebra

$\overset{\text{def}}{\iff}$

- A : graded module
- $m = \{m_n\}_{n\geq 1}$
  $m_n : A^{\otimes n} \to A$: linear map of deg $(n-2)$
  satisfying

$$(*_n)\quad \sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r}(1^{\otimes p}\otimes m_q \otimes 1^{\otimes r}) = 0$$

meaning of m_n

$n=1$ $m_1 : A \to A$: linear map of deg(-1)

$(*_1)$ $m_1 m_1 = 0$

$\rightsquigarrow (A, m_1)$: chain complex

$n=2$ $m_2 : A^{\otimes 2} \to A$: linear map of deg 0

$(*_2)$ $m_1m_2 - m_2(m_1\otimes 1) - m_2(1\otimes m_1) = 0$

$\iff m_1m_2 = m_2(m_1\otimes 1 + 1\otimes m_1)$

$(d\mu = \mu(d\otimes 1 + 1\otimes d))$

$\rightsquigarrow$ m_1 is a derivation w.r.t. the "multiplication" m_2

$n=3$ $m_3 : A^{\otimes 3} \to A$: linear map of deg $(+1)$

$(*_3) \iff m_2(1\otimes m_2) - m_2(m_2\otimes 1)$
$= m_1m_3 + m_3(m_1\otimes 1\otimes 1 + 1\otimes m_1\otimes 1 + 1\otimes 1\otimes m_1)$

$\rightsquigarrow$ The "multiplication" m_2 is associative up to homotopy m_3

$\boxed{A_\infty\text{-algebra} = \text{"homotopy associative algebra"}}$

Ex1.5

- (A, μ, d) : dg algebra
  ⇝ (A, m) : A_∞-algebra is defined by
  $m_1 := d,\ m_2 := \mu,\ m_n = 0\ (n \geq 3)$
- X : A_∞-space
  ⇝ $C_*^{sing}(X)$: A_∞-algebra

Why A_∞-algebra?

dg algebra has some problems

Problem1.6

In general, a quasi-isomorphism of dg algebras does NOT have its "inverse".

i.e.

$f : (A, \mu^A, d^A) \xrightarrow{\simeq} (B, \mu^B, d^B)$
: quasi-isom of dg algebras
$(H(f) : H(A, d^A) \xrightarrow{\cong} H(B, d^B)$: isom$)$

$\cancel{\Rightarrow}\ \exists g : (B, \mu^B, d^B) \longrightarrow (A, \mu^A, d^A)$
: quasi-isom of dg algebras
s.t. $H(g) = H(f)^{-1}$

Problem1.7

In general, the information of a dg algebra (A, μ, d) can NOT be recovered from its homology $(H(A), H(\mu))$.

Ex1.8 (for Problem1.6)

$(A, \mu^A, d^A) := (\mathbb{Q}[x] \otimes \Lambda(y), \mu, d)$ (polynomial alg, exterior alg)

where
- $|x| = 2n,\ |y| = 4n+1$
- d is defined by $dx = 0,\ dy = x^2$ (and Leibniz rule)

$((A, \mu^A, d^A)$ is quasi-isom to $A^*_{dR}(S^{2n}))$

$B := H(A, d^A) = \mathbb{Q}\{[1], [x]\}$: 2-dim
$\mu^B := H(\mu^A),\ d^B := 0$

$$f : (A, \mu^A, d^A) \longrightarrow (B, \mu^B, d^B)$$
$$x^k \longmapsto \begin{cases} [x^k] & (k = 0, 1) \\ 0 & (k \geq 2) \end{cases}$$
$$x^k y \longmapsto 0$$

⇝ f is quasi-isom

But g does NOT exist

∵ $g : (H(A), H(\mu^A)) \to (A, \mu^A)$: graded alg hom
$\Rightarrow \exists \lambda \in \mathbb{Q},\ g([x]) = \lambda x$
$0 = g([x]^2) = g([x])^2 = \lambda^2 x^2$
$\therefore \lambda = 0,\ g([x]) = 0$ ⇝ $H(g)([x]) = 0$
g : NOT quasi-isom

To construct

$$g : (B, \mu^B, d^B) \longrightarrow (A, \mu^A, d^A),$$

we need

	(B, μ^B, d^B)	(A, μ^A, d^A)
	free	cofree
	projective	injective
	cofibrant	fibrant

Ex1.9 (for Problem1.7)

$(A, \mu, d) = (\Lambda(x, y, z), \mu, d)$
(where $|x| = 2n-1,\ |y| = 2n+1,\ |z| = 4n+1$
$dx = 0,\ dy = 0,\ dz = xy$)

Then

$(A, \mu, d) \not\simeq (H(A), H(\mu), 0)$
: NOT quasi-isom

⇝ These are homotopically different.

∵ Assume these are quasi-isom.
Then, since (A, μ) is free as graded commutative algebra,
$\exists f : (A, \mu, d) \xrightarrow{\simeq} (H(A), H(\mu), 0)$: quasi-isom
But
$f(z) = 0 \in H_{4n+1}(A) = 0$
⇝ $f(xyz) = 0$
$\underbrace{H(f)([xyz])}_{\neq 0} = 0$ contradict

§2. Coalgebra

A_∞-algebra $=$ $\begin{pmatrix}\text{cofree graded coalgebra} \\ + \text{differential}\end{pmatrix}$

Def 2.1

- (C, Δ) : <u>graded coalgebra</u>

$\overset{\Longleftrightarrow}{\text{def}}$
- C : graded module
- $\Delta : C \to C \otimes C$: linear map of deg 0 satisfying coassociativity : $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

- $d : C \to C$: <u>coderivation of deg k</u>

$\overset{\Longleftrightarrow}{\text{def}}$
- $d : C \to C$: linear map of deg k
- $\Delta d = (1 \otimes d + d \otimes 1)\Delta$

Def 2.2

V : graded module

Define

$(\overline{T}^c V, \Delta)$: <u>(reduced) tensor coalgebra</u>

by
- $\overline{T}^c V = \bigoplus_{n \geq 1} V^{\otimes n}$ as graded module
  $v_1 \otimes v_2 \otimes \cdots \otimes v_n = v_1 v_2 \cdots v_n$
- $\Delta : \overline{T}^c V \longrightarrow \overline{T}^c V \otimes \overline{T}^c V$
  $v_1 \cdots v_n \longmapsto \sum_{1 \leq i \leq n-1} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$

Denote

$p : \overline{T}^c V \longrightarrow V$: the projection

Prop 2.3

The tensor coalg $(\overline{T}^c V, \Delta)$ is cofree on V in the category of "conilpotent" coalgebra

$$\begin{tikzcd} & \overline{T}^c V \arrow[d, "p"] \\ C \arrow[ur, dashed, "\exists ! f"] \arrow[r, "\forall \bar{f}"'] & V \end{tikzcd} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{tikzcd} & \overline{T} V \arrow[dl, dashed, "\exists ! g"'] \\ A & V \arrow[l, "\forall \bar{g}"] \arrow[u, "i"'] \end{tikzcd}$$

coalg hom.

$\forall \bar{f}$: linear map of deg 0

$$\begin{cases} \cdot\ \bar{f} = pf \\ \cdot\ f = \sum_{n \geq 1} \bar{f}^{\otimes n} \Delta^{n-1} \end{cases} \overset{1:1}{\longleftrightarrow} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{cases} \cdot\ \bar{g} = gi \\ \cdot\ g = \sum_{n \geq 1} \mu^{n-1} \bar{g}^{\otimes n} \end{cases}$$

$(\bar{g}(v_1) \cdots \bar{g}(v_n))$

(<u>where</u> Δ^{n-1} is defined by $\Delta^0 = 1$, $\Delta^i = (\Delta \otimes 1 \otimes \cdots \otimes 1)\Delta^{i-1}$)

Cor 2.4

V, W : graded module

$f : (\overline{T}^c V, \Delta) \longrightarrow (\overline{T}^c W, \Delta)$: graded coalg. hom

$\overset{1:1}{\longleftrightarrow}$ $\bar{f} : \overline{T}^c V \longrightarrow W$: linear map of deg 0

$\overset{1:1}{\longleftrightarrow}$ $\{\bar{f}_n : V^{\otimes n} \longrightarrow W$: linear map of deg 0$\}_{n \geq 1}$

$$f|_{V^{\otimes m}} = \sum_{n \geq 1} \sum_{i_1 + \cdots + i_n = m} \bar{f}_{i_1} \otimes \cdots \otimes \bar{f}_{i_n}$$

similar for coderivations

Prop 2.5

V : graded module, $k \in \mathbb{Z}$

$d : \overline{T}^c V \longrightarrow \overline{T}^c V$: coderivation of deg k

$\overset{1:1}{\longleftrightarrow}$ $\bar{d} : \overline{T}^c V \to V$: linear map of deg k

$\overset{1:1}{\longleftrightarrow}$ $\{\bar{d}_n : V^{\otimes n} \longrightarrow V$: linear map of deg $k\}_{n \geq 1}$

$$d|_{V^{\otimes m}} = \sum_{p+q+r=m} 1^{\otimes p} \otimes \bar{d}_q \otimes 1^{\otimes r} \quad \text{— (A)}$$

§3. A_∞-algebra

Def 3.1 (A_∞-algebra)

(A, m) : A_∞-algebra

$\overset{\text{def}}{\Longleftrightarrow}$
- A : graded module
- $m : \bar{T}^c(sA) \to \bar{T}^c(sA)$: coderivation of deg -1 s.t. $m^2 = 0$

(where sA : graded mod. defined by $(sA)_n = A_{n-1}$)

Lem 3.2

Def 3.1 is equivalent to Def 1.4

proof By Prop 2.5,

$m : \bar{T}^c(sA) \to \bar{T}^c(sA)$: coderivation of deg(-1)

$\overset{1:1}{\longleftrightarrow} \{\bar{m}_n : (sA)^{\otimes n} \to sA$: linear map of deg$(-1)\}_{n \geq 1}$

$\overset{1:1}{\longleftrightarrow} \{m_n : A^{\otimes n} \to A$: linear map of deg$(n-2)\}_{n \geq 1}$

Write m^2 in terms of m_n by ⊛

$\rightsquigarrow m^2 = 0 \Leftrightarrow (\forall n \ (*_n))$ //

Rmk 3.3

If (A, m) is constructed from a dg alg, the dg coalgebra $(\bar{T}^c(sA), \Delta, m)$ is the usual bar construction.

$\rightsquigarrow$ some kind of "classifying space"

Def 3.4 (∞-morphism)

$(A, m^A), (B, m^B)$: A_∞-algebras

$f : (A, m^A) \rightsquigarrow (B, m^B)$: $\overset{(A_\infty\text{-})}{\infty\text{-morphism}}$

$\overset{\text{def}}{\Longleftrightarrow} f : (\bar{T}^c(sA), \Delta, m^A) \to (\bar{T}^c(sB), \Delta, m^B)$: dg coalg hom

Lem 3.5

Def 3.4 is equivalent to the following:

$f_n : A^{\otimes n} \to B$: linear map of deg $n-1$

satisfying

$$(\#_n) \quad \sum_{p+q+r=n} (-1)^{p+qr} f_{p+1+r}(1^{\otimes p} \otimes m_q \otimes 1^{\otimes r}) = \sum_{1 \leq \ell \leq n} \sum_{i_1 + \cdots + i_\ell = n} (-1)^\varepsilon m^B_\ell (f_{i_1} \otimes \cdots \otimes f_{i_\ell})$$

(where $\varepsilon = (\ell-1)(i_1 - 1) + (\ell-2)(i_2-1) + \cdots + 1 \cdot (i_{\ell-1} - 1)$)

meaning of f_n

$\underline{n=1}$ $f_1 : (A, m_1^A) \to (B, m_1^B)$: chain map (of deg 0)

$\underline{n=2}$ f_1 preserves multiplication up to homotopy f_2

Ex 3.6

$f : (A, \mu^A, d^A) \to (B, \mu^B, d^B)$: dg alg hom

$\rightsquigarrow f : (A, m^A) \rightsquigarrow (B, m^B)$: ∞-morph

defined by $\begin{cases} f_1 = f : A \to B \\ f_n = 0 : A^{\otimes n} \to B \ (n \geq 2) \end{cases}$

composition of ∞-morphisms

$(A, m^A) \overset{f}{\rightsquigarrow} (B, m^B) \overset{g}{\rightsquigarrow} (C, m^C)$: ∞-morph's

Their composition is defined by the composition

$$gf : \bar{T}^c(sA) \xrightarrow{f} \bar{T}^c(sB) \xrightarrow{g} \bar{T}^c(sC)$$

Then

$$(gf)_n = \sum_{1 \leq \ell \leq n} \sum_{i_1 + \cdots + i_\ell = n} (-1)^\varepsilon g_\ell (f_{i_1} \otimes \cdots \otimes f_{i_\ell})$$

Def 3.7 (homotopy)

$f, g : (A, m^A) \rightsquigarrow (B, m^B)$: ∞-morph's

h : homotopy from f to g

$\overset{\text{def}}{\Longleftrightarrow}$ Consider $f, g : (\bar{T}^c(sA), \Delta, m^A) \to (\bar{T}^c(sB), \Delta, m^B)$
- $h : \bar{T}^c(sA) \to \bar{T}^c(sB)$: (f,g)-coderivation of deg$(+1)$
- $f - g = m^B h + h m^A$

$\Delta h = (f \otimes h + h \otimes g) \Delta$

Lem 3.8

Def 3.7 is equivalent to the following:

$h_n : A^{\otimes n} \to B$: linear map of deg n

satisfying

$$f_n - g_n = \sum_{p+q+r=n} (-1)^{p+qr} h_{p+1+r}(1^{\otimes p} \otimes m^A_q \otimes 1^{\otimes r}) + \sum_{k \geq 1} m^B_k \sum_{\substack{i_1 + \cdots + i_\ell + q \\ + j_1 + \cdots + j_{k-\ell-1} = n}} (-1)^\varepsilon f_{i_1} \otimes \cdots \otimes f_{i_\ell} \otimes h_q \otimes g_{j_1} \otimes \cdots \otimes g_{j_{k-\ell-1}}$$

h_1 is a homotopy from f_1 to g_1 as chain maps.

§4. Advantage of A_∞-algebras to dg algebras

Def 4.1

$f : (A, m^A) \rightsquigarrow (B, m^B)$: ∞-morph

f : quasi-isom

$\underset{\text{def}}{\Leftrightarrow}$ $f_1 : (A, m_1^A) \to (B, m_1^B)$

: quasi-isom of complexes

Thm 4.2

$f : (A, m^A) \rightsquigarrow (B, m^B)$: quasi-isom

$\Rightarrow$ f : homotopy equivalence

(i.e. $\exists g : (B, m^B) \rightsquigarrow (A, m^A)$: ∞-morph

s.t. $fg \simeq 1_B$, $gf \simeq 1_A$)

sketch of proof

f is equivalent to

$f : (\overline{T}^c(sA), \Delta, m^A) \to (\overline{T}^c(sB), \Delta, m^B)$

: quasi-isom of dg coalg

Since $\overline{T}^c(sA)$ is cofree,

$\exists g : (\overline{T}^c(sB), \Delta, m^B) \to (\overline{T}^c(sA), \Delta, m^A)$

: "inverse" of f. //

$\hookrightarrow$ Problem 1.6

(Consider the case (A, m^A), (B, m^B) come from dg algebras)

Thm 4.3

Define two categories by

DGA : Obj dg algebras

Mor dg algebra homomorphisms

DGA$'$: Obj dg algebras

Mor ∞-morphisms

Then

$\mathrm{DGA}[\text{quasi-isom}^{-1}] \xrightarrow{\simeq} \mathrm{DGA}'/_{\simeq}$

(invert quasi-isom's

morph: $A \xleftarrow{\simeq} C \to B$)

($\simeq$: homotopy

morph: $\infty\text{-Morph}(A,B)/_{\simeq}$)

Thm 4.5

(A, m) : A_∞-algebra

Then

$HA = H(A, m_1)$ has an A_∞-algebra structure $(HA, \overline{m})$ s.t.

- $\overline{m}_1 = 0$, $\overline{m}_2 = H(m_2)$
- $(A, m) \simeq (HA, \overline{m})$: homotopy equivalent

This structure is unique up to ∞-isom.

$\hookrightarrow$ Problem 1.7

References

- Loday-Vallette, Algebraic operads, Springer (Chapter 9, 10)
- Lefèvre, Sur les A-infini catégories, arXiv:0310337 (Chapter 1)