代数的トポロジー入門
– ホモトピー群の光と闇 –

多元入門シリーズ [NISS]

若月駿　2026 年 10 月

このノートは書きかけなので，不完全な記述が多数あります．下記ページから最新版を取得してください．

https://shwaka.github.io/ja/heavy/notes/NagoyaIntroSeminar.pdf

概要

ホモトピー群は代数的トポロジーにおける主要な研究対象のひとつである．本講演では，ホモトピー群が非常に複雑で計算が難しいという「闇」と，その中にある規則性や代数的計算方法などの「光」を紹介する．また，その過程で suspension, based loop space, fibration, homotopy fiber などの，代数的トポロジーにおける基本的かつ重要な概念についても (鎖複体の場合との違いにも少し触れつつ) 説明する．さらに (時間が許せば) 有理ホモトピー論についても解説し，有理数体係数であればホモトピー群の計算がかなり簡単になることを説明したい．

1 回目の講演については特に専門知識を仮定せず，位相空間や (有限生成) アーベル群の基礎事項のみを前提知識とする．2 回目以降は基本的なホモロジー代数 (特に長完全列の扱い) にある程度慣れていることを仮定する．また，被覆空間やファイバー束についても定義程度を知っていることが望ましいが，必要に応じて補足説明を行う．

参考文献について

[Hat02] 言わずと知れた入門書．著者自身がウェブサイトで PDF を公開している．本ノートの多くの部分は Chapter 4 に書かれている．

[Hat] 上の本の「続編」と書かれた，スペクトル系列の PDF．まだ執筆途中のようだが，長いこと更新されてなさそう．複数のバージョンがあるが，上の本の続きとして Chapter 5 となっているものを参照する．本ノートの第 4 節と第 6 節は概ねこの PDF に書かれている．

[服部 91]

目次

1 Introduction 2

2 Definition of homotopy groups 3
2.1 Definition of $\pi_k(X)$ as a set . . . 3
2.2 Group structure of $\pi_k(X)$ . . . 3

3 Properties of homotopy groups (of spheres) 5
3.1 Covering space . . . 5
3.2 Hurewicz isomorphism . . . 5
3.3 Suspension isomorphism . . . 6
3.4 Σ-Ω adjunction and proof of 定理 3.14 . . . 7
3.5 Whitehead theorem . . . 7

4 Serre classes 9
4.1 Serre classes and Hurewicz isomorphism . . . 9
4.2 Finite generation of $\pi_k(S^n)$ . . . 10
4.3 Torsions in $\pi_k(S^n)$ . . . 10
4.4 Localization of spaces and exponents of homotopy groups . . . 11

5 Fibration 13
5.1 Fibration and homotopy groups . . . 13
5.2 Fibration and cohomology groups . . . 13
5.3 Homotopy fiber . . . 15
5.4 Mapping cone and cylinder . . . 16

6 Killing homotopy の方法 18
6.1 Eilenberg-MacLane space . . . 18
6.2 Cartan-Serre's killing homotopy method . . . 18

7 Rational homotopy theory 21
7.1 Sullivan model . . . 21
7.2 Formality . . . 21
7.3 Rational homotopy groups . . . 21

References 22

1 Introduction

Aims of algebraic topology:

- Distinguish spaces via (algebraic) invariants
  - (co)homology, homotopy group, K-theory, ...etc
- Study the homotopy set $\mathrm{Map}(X,Y)/\simeq$

$\mathrm{Map}(X,Y)/\simeq$

- $\mathrm{Map}(X,Y) = \mathrm{Hom}_{\mathrm{Top}}(X,Y)$: the morphism set is "too large"
- $\mathrm{Map}(X,Y)/\simeq$ is much smaller, but still difficult
- $\pi_n(X) := \mathrm{Map}(S^n,X)/\simeq$: homotopy group is the simplest case (but still difficult)
- Other important examples:
  - $H^n(X;A) \cong \mathrm{Map}(X,K(A,n))/\simeq$
  - $\{\text{principal } G\text{-bundle over } X\} \cong \mathrm{Map}(X,BG)/\simeq$

(Co)homology and homotopy groups

- (co)homology
  - easy to compute
- homotopy group
  - hard to compute
  - Y: CW complex
    ↑ space constructed by gluing disks (e.g. mfd)

    ⇝ $\pi_*(X)$ contains "info" on maps $f\colon Y \to X$

問題 1.2.

(1) What properties does $\pi_k(X)$ have?

(2) How can we compute $\pi_k(X)$, especially $\pi_k(S^n)$?

例 1.3 (Homology groups of spheres).

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	0	0	0	0	0	0	0
S^3	0	0	$\mathbb{Z}$	0	0	0	0	0	0
S^4	0	0	0	$\mathbb{Z}$	0	0	0	0	0
S^5	0	0	0	0	$\mathbb{Z}$	0	0	0	0
S^6	0	0	0	0	0	$\mathbb{Z}$	0	0	0

Easily computed by using Mayer-Vietoris exact seq:

$$H_k(S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k = n, 0) \\ 0 & (\text{otherwise}) \end{cases}$$

例 1.4 (Homotopy groups of spheres, [Wikc]).

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

- Very difficult!!!
- See 表 1.1 for a larger table

Some patterns:

- $\forall k > 1,\ \pi_k(S^1) = 0$ (系 3.2)
- $k \le n \implies \pi_k(S^n) \cong H_k(S^n)$
  - $k < n \implies \pi_k(S^n) = 0$ (系 3.8 (1))
  - $\pi_n(S^n) \cong \mathbb{Z}$ (系 3.8 (2))
- $\pi_k(S^n) \cong \pi_{k+1}(S^{n+1})$ for some range (系 3.15)
- No p-torsion for a large prime p (≥ 5)? (定理 4.14, 定理 4.15)
- $\pi_k(S^n)$ is finitely generated for any (k,n) (系 4.10)
- $\pi_k(S^n)$ is finite for most (k,n) (定理 4.12)

Notation

- $\simeq$: homotopic, homotopy equivalence
- $\cong$: isomorphic, homeomorphic
- $\sim$: an equivalence relation
- $\mathbb{N} = \{0,1,2,\dots\}$
- For (non-empty) $A \subset X$, write
  $X/A := X/\sim$ ← A を一点に潰した空間

  where $x \sim x' \overset{\text{def}}{\iff} x \in A$ and $x' \in A$

表 1.1: Some (unstable) homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/12\oplus\mathbb{Z}/2$	$\mathbb{Z}/84\oplus(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/12\oplus\mathbb{Z}/2$	$\mathbb{Z}/84\oplus(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24\oplus\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/120\oplus\mathbb{Z}/12\oplus\mathbb{Z}/2$	$\mathbb{Z}/84\oplus(\mathbb{Z}/2)^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/72\oplus\mathbb{Z}/2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/24\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$	$(\mathbb{Z}/2)^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/120$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$

2 Definition of homotopy groups

In this section, we

- give precise definition of homotopy groups
- prove some basic properties

2.1 Definition of $\pi_k(X)$ as a set

定義 2.1.

- (X, x_0): based space
  $\overset{\text{def}}{\iff}$ ∘ X: top. space
  ∘ $x_0 \in X$
- $f\colon (X, x_0) \to (Y, y_0)$: base point preserving map
  $\overset{\text{def}}{\iff} f(x_0) = y_0$
- $\mathrm{Map}_*((X, x_0), (Y, y_0))$: the set of base point preserving maps
- $f, g\colon (X, x_0) \to (Y, y_0)$: base point preserving maps
  Define $f \simeq g$: homotopic
  $\overset{\text{def}}{\iff} \exists H\colon X \times I \to Y$ s.t.
  ∘ $H(-, 0) = f$
  ∘ $H(-, 1) = g$
  ∘ $\forall t \in I,\ H(x_0, t) = y_0$

定義 2.2.

$k \in \mathbb{N}$ ← possibly $k = 0$

Define

- $S^k \coloneqq \left\{ (x_0, x_1, \ldots, x_k) \;\middle|\; \sum_i (x_i)^2 = 1 \right\} \subset \mathbb{R}^{k+1}$
  the unit sphere
- $s_0 \coloneqq (1, 0, 0, \ldots, 0) \in S^k$: base point
  $\rightsquigarrow$ $S^k = (S^k, s_0)$: based space

定義 2.3.

$k \in \mathbb{N}$
$X = (X, x_0)$: based space

Define

$$\pi_k(X, x_0) \coloneqq \mathrm{Map}_*(S^k, X)/\simeq$$

(x_0 is omitted sometimes)

2.2 Group structure of $\pi_k(X)$

定義 2.4.

(X, A): pair of spaces (i.e. $A \subset X$)
(Y, B): pair of spaces (i.e. $B \subset Y$)

Define

$$\mathrm{Map}((X, A), (Y, B)) \coloneqq \{f\colon X \to Y \mid f(A) \subset B\} \subset \mathrm{Map}(X, Y)$$

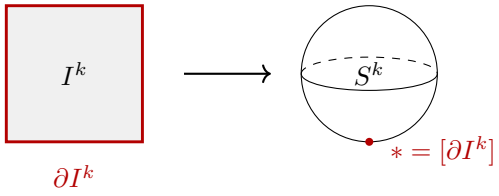
補題 2.5.

$$\pi_k(X, x_0) \cong \mathrm{Map}((I^k, \partial I^k), (X, x_0))/\simeq$$

where

- $I^k = [0, 1]^k$: k-dim cube
- ∂I^k: boundary of I^k

証明 $I^k/\partial I^k \cong S^k$:



□

注意 2.6.

Note on π_0:

- $S^0 = \{x \in \mathbb{R} \mid |x| = 1\} = \{\pm 1\}$
  $s_0 = +1 \in S^0$
  $\rightsquigarrow$ $\mathrm{Map}_*(S^0, X)$
  $= \mathrm{Map}((\{\pm 1\}, +1), (X, x_0))$
  $= \mathrm{Map}(\{+1\}, X)$
  $\cong X$
- $I^0 = \{*\}$ (one element)
  $\partial I^0 = \emptyset$
  $\rightsquigarrow$ $\mathrm{Map}((I^0, \partial I^0), (X, x_0))$
  $= \mathrm{Map}((\{*\}, \emptyset), (X, x_0))$
  $= \mathrm{Map}(\{*\}, X)$
  $\cong X$

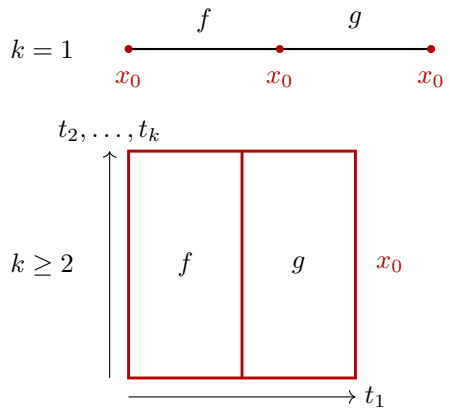
Hence

- 補題 2.5 holds also for $k = 0$
- $\pi_0(X) =$ (path components of X)

定義 2.7.

$k \geq 1$
(X, x_0): based space
$f, g \in \mathrm{Map}((I^k, \partial I^k), (X, x_0))$
Define $f \cdot g \in \mathrm{Map}((I^k, \partial I^k), (X, x_0))$ by

$$(f \cdot g)(t_1, t_2, \ldots, t_k) \coloneqq \begin{cases} f(2t_1, t_2, \ldots, t_k) & (0 \leq t_1 \leq \frac{1}{2}) \\ g(2t_1 - 1, t_2, \ldots, t_k) & (\frac{1}{2} \leq t_1 \leq 1) \end{cases}$$



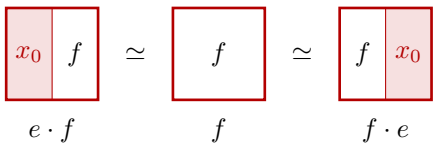
補題 2.8.

(1) $k \geq 1 \implies \pi_k(X)$: group
(2) $k \geq 2 \implies \pi_k(X)$: abelian group

証明

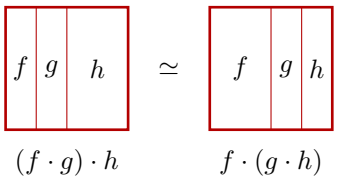
(1) Let $e\colon I^k \to X$ be the constant map at x_0.

Unitality



$\rightsquigarrow [e] \cdot [f] = [f] = [f] \cdot [e]$

Associativity

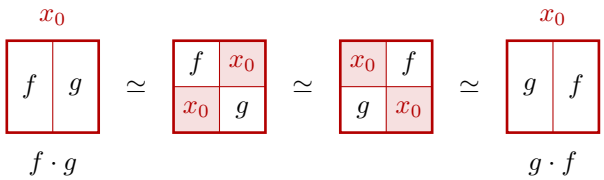


$\rightsquigarrow ([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$

Inverse Define $\bar{f}(t_1, t_2, \dots, t_k) \coloneqq f(1 - t_1, t_2, \dots, t_k)$

$\rightsquigarrow f \cdot \bar{f} \simeq e \simeq \bar{f} \cdot f$

(2)



□

Generalizing the proof of 補題 2.8 (2), we have:

補題 2.9 (Eckmann-Hilton argument [EH62, Theorem 4.17])**.**

X: set
$\circ\colon X \times X \to X$, $1_\circ$: unit for $\circ$
$\star\colon X \times X \to X$, $1_\star$: unit for $\star$

Assume

$\forall a, b, c, d \in X, \quad (a \circ b) \star (c \circ d) = (a \star c) \circ (b \star d)$

Then

(1) $1_\circ = 1_\star$
(2) $\circ = \star$
(3) $\circ$ is associative
(4) $\circ$ is commutative

証明

(1) $a = d = 1_\circ, \quad b = c = 1_\star$

($\rightsquigarrow$ Write $1 \coloneqq 1_\circ = 1_\star$ for simplicity)

(2) $b = c = 1$

(3) $b = 1$

(4) $a = d = 1$

□

注意 2.10.

(1) Write

- $a \circ b = \begin{pmatrix} a & b \end{pmatrix}$
- $a \star c = \begin{pmatrix} a \\ c \end{pmatrix}$

Then the assumption implies

$$\begin{pmatrix} (a & b) \\ (c & d) \end{pmatrix} = \left(\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \right)$$

$\rightsquigarrow$ We can write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ without ambiguity

$\rightsquigarrow$ Similar picture as in 補題 2.9 (4)

(2) Roughly speaking (assuming associativity), 補題 2.9 states:

X: monoid object in the category of monoids
$\implies$ X: commutative monoid

注意 2.11.

In 定義 2.7, $f \cdot g$ is defined by using the first param t_1. By 補題 2.9 (2), choosing another t_i does not change the group structure.

3 Properties of homotopy groups (of spheres)

In this section, we explain:

- isom for covering spaces
- Hurewicz isom
- suspension isom
- Whitehead theorem (importance of π_*)

3.1 Covering space

See [Hat02, Section 1.3] for definition of covering spaces.

定理 3.1.

B: path-connected space

$p: E \to B$: covering space

Then

$\forall k > 1, \quad p_*: \pi_k(E) \xrightarrow{\cong} \pi_k(B)$: isom

証明 Recall ([Hat02, Prop 1.33]):

$$\begin{array}{ccc} & & E \\ & \overset{\exists}{\nearrow} & \downarrow p \\ X & \xrightarrow{f} & B \end{array} \iff f_*(\pi_1(X)) \subset p_*(\pi_1(E))$$

Surjectivity Since $\pi_1(S^k) = 0$ (↑ by $k > 1$), every map $S^k \to B$ has a lift:

$$\begin{array}{ccc} & & E \\ & \overset{\exists}{\nearrow} & \downarrow p \\ S^k & \longrightarrow & B \end{array}$$

Injectivity ← also holds for $k = 1$

Given $[g] \in \operatorname{Ker} p_*$

i.e. $g: S^k \to E$ with $p \circ g \simeq \text{const}$

$\rightsquigarrow \exists h: D^{k+1} \to B$ s.t. $h|_{S^k} = g$

Since $\pi_1(D^{k+1}) = 0$, we have a lift:

$$\begin{array}{ccc} & \overset{\exists \tilde{h}}{\nearrow} & E \\ & & \downarrow p \\ D^{k+1} & \xrightarrow{h} & B \end{array}$$

$\rightsquigarrow \tilde{h}|_{S^k} = g$

$\rightsquigarrow [g] = 0 \in \pi_1(E)$ □

系 3.2.

$\forall k > 1, \quad \pi_k(S^1) = 0$

証明 Apply 定理 3.1 to the universal covering $p: \mathbb{R} \to S^1$ □

3.2 Hurewicz isomorphism

定義 3.3 (n-connected).

$n \in \mathbb{N}$

X: top. space

Define

X: n-connected

$\overset{\text{def}}{\iff} 0 \le \forall k \le n, \quad \pi_k(X) = 0$

注意 3.4.

- 0-connected = path-connected
- 1-connected = simply connected

定義 3.5 (Hurewicz map).

$k \ge 1$

X: top. space

Define

$$\begin{array}{rccc} \text{Hur}: & \pi_k(X) & \longrightarrow & H_k(X;\mathbb{Z}) \quad : \text{grp hom} \\ & (f: S^k \to X) & \longmapsto & f_*[S^k] \end{array}$$

where

$[S^k] \in H_k(S^k;\mathbb{Z})$: the generator of $H_k(S^k;\mathbb{Z}) \cong \mathbb{Z}$ (↑ fundamental class)

命題 3.6.

X: 0-connected (i.e. path-connected)

Then Hur: $\pi_1(X) \to H_1(X;\mathbb{Z})$ induces

$$\pi_1(X)^{\text{ab}} \xrightarrow{\cong} H_1(X;\mathbb{Z})$$

(↓ abelianization of a group)

証明 See [Hat02, Thm 2A.1] □

定理 3.7 (Hurewicz isom (Hurewicz 1935 [Hur35])).

$n \ge 2$

X: $(n-1)$-connected space

Then

(1) $\forall k \le n-1, \ H_k(X;\mathbb{Z}) = 0$ (↑ or $\forall$ ab. group)

(2) Hur: $\pi_n(X) \xrightarrow{\cong} H_n(X;\mathbb{Z})$: isom

証明 See [Hat02, Thm 4.32] □

系 3.8.

$n \ge 2$

Then

(1) $\forall k < n, \ \pi_k(S^n) = 0$

(2) $\pi_n(S^n) \cong \mathbb{Z}$

証明

- $\pi_1(S^n) = 0$

  [$\because$ van Kampen theorem

- $\forall k < n, \ H_k(X;\mathbb{Z}) = 0$

$\rightsquigarrow$ By induction on k, we have (↑ 命題 3.6, 定理 3.7)

$\forall k \le n$, Hur: $\pi_k(X) \xrightarrow{\cong} H_k(X;\mathbb{Z})$ □

系 3.8 の別証明 [河澄 22, 定理 5.1.7, 定理 5.1.8] **(1)**

Given $[f] \in \pi_k(S^n)$

⇝ Approximate f by a C^∞ map

⇝ f: non-surjective

↑ Sard Theorem, $k < n$

⇝ f is null-homotopic

↑ $S^k \setminus \{*\} \cong \mathbb{R}^k$: contractible

(2) Mapping degree gives an isom $\pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ □

There is also a relative version of Hurewicz isomorphism

定理 3.9 (relative Hurewicz isom)**.**

$n \ge 2$

(X, A): $(n-1)$-connected pair

↑ i.e. $\forall k \le n-1,\ \pi_k(X, A) = 0$

A: 1-connected

Then

(1) $\forall k \le n-1,\ H_k(X, A;\mathbb{Z}) = 0$

(2) Hur: $\pi_n(X, A) \xrightarrow{\cong} H_n(X, A;\mathbb{Z})$

証明 See [Hat02, Thm 4.32] □

系 3.10.

X, Y: 1-conn spaces

$f: X \to Y$

$n \ge 2$

Assume

$$f_*: H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z}): \begin{cases} \text{isom} & (k < n) \\ \text{surj} & (k = n) \end{cases}$$

Then

$$f_*: \pi_k(X) \to \pi_k(Y): \begin{cases} \text{isom} & (k < n) \\ \text{surj} & (k = n) \end{cases}$$

証明

Case 1 $X \subset Y$: subspace

By assump and long exact seq for H_*,

$\forall k \le n,\ H_k(Y, X;\mathbb{Z}) = 0$

⇝ $\forall k \le n,\ \pi_k(Y, X) = 0$

↑ 定理 3.9

⇝ $f_*: \pi_k(X) \to \pi_k(Y)$: $\begin{cases} \text{isom} & (k < n) \\ \text{surj} & (k = n) \end{cases}$

↑ long exact seq for π_*

Case 2 General

Replace $f: X \to Y$ with its mapping cylinder

↑ See 定義 5.17

⇝ We can apply Case 1. □

3.3 Suspension isomorphism

定義 3.11.

(X, x_0): based top. space

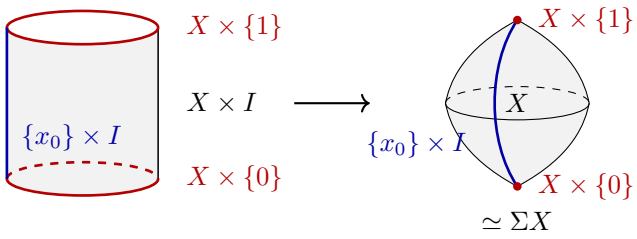
Define

$\Sigma X \coloneqq (X \times I)/\sim$: (reduced) suspension of X

$\cong (X \times I)/((X \times \partial I) \cup (\{x_0\} \times I))$

where $\sim$ is generated by

- $\forall x, x' \in X,\ (x, 0) \sim (x', 0)$
- $\forall x, x' \in X,\ (x, 1) \sim (x', 1)$
- $\forall t, t' \in I,\ (x_0, t) \sim (x_0, t')$ ← これを取り除くと unreduced suspension



補題 3.12.

$k \in \mathbb{N}$

Then

$$\Sigma S^k \cong S^{k+1}$$

証明 Regard $S^k = I^k/\partial I^k$

⇝ $\Sigma S^k \cong (S^k \times I)/((S^k \times \partial I) \cup (\{s_0\} \times I))$

$\cong (I^k \times I)/((I^k \times \partial I) \cup (\partial I^k \times I))$

$\cong I^{k+1}/\partial I^{k+1}$

$\cong S^{k+1}$ □

定義 3.13.

Define the suspension homomorphism E by

$$E: \begin{array}{ccc} \pi_k(X) & \longrightarrow & \mathrm{Map}_*(\Sigma S^k, \Sigma X)/\simeq \ \cong \pi_{k+1}(\Sigma X) \\ f & \longmapsto & \Sigma f \end{array}$$

定理 3.14 (Freudenthal suspension theorem [Fre37], [BM51])**.**

↑ for S^n

↑ for general X, by excision thm

$n \ge 2$

X: $(n-1)$-connected space

Then The map $E: \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is

- isom for $\forall k < 2n-1$
- surj for $k = 2n-1$

Proof will be given later.

系 3.15.

(1) $\forall k < 2n-1$, $\pi_k(S^n) \xrightarrow{\cong} \pi_{k+1}(S^{n+1})$: isom

(2) $\pi_{2n-1}(S^n) \twoheadrightarrow \pi_{2n}(S^{n+1})$: surj

注意 3.16.

For each $m \in \mathbb{N}$, we have a seq

$$\pi_m(S^0) \to \pi_{m+1}(S^1) \to \cdots \to \pi_{m+n}(S^n) \to \cdots$$
$$\to \pi_{2m+1}(S^{m+1}) \twoheadrightarrow \pi_{2m+2}(S^{m+2}) \xrightarrow{\cong} \pi_{2m+3}(S^{m+3})$$
$$\xrightarrow{\cong} \pi_{2m+4}(S^{m+4}) \xrightarrow{\cong} \cdots$$

$\because$ Surj and isom follow from 系 3.15

$\rightsquigarrow$ $\pi_{m+n}(S^n)$ stabilizes for $n \geq m+2$

This is called the m-th stable homotopy group of spheres (isom to $\operatorname{colim}_n \pi_{m+n}(S^n)$)

3.4 Σ-Ω adjunction and proof of 定理 3.14

定義 3.17.

$\Omega X \coloneqq \operatorname{Map}_*(S^1, X)$

↑ with the compact-open topology (See [Hat02, Appendix, p.529])

命題 3.18.

$\Sigma\colon \mathrm{Top}_* \rightleftharpoons \mathrm{Top}_* : \Omega$ is an adjunction

In other words,

$$\operatorname{Map}_*(\Sigma X, Y) \xrightarrow[\text{ad}]{\cong} \operatorname{Map}_*(X, \Omega Y)$$

証明 Follows from

- $X \times I \twoheadrightarrow \Sigma X$: quotient
- $\Omega X \subset X^I$: subspace
- $\operatorname{Map}(X \times I, Y) \cong \operatorname{Map}(X, Y^I)$

□

補題 3.19.

$\pi_k(\Omega X) \cong \pi_{k+1}(X)$

証明 Follows from 命題 3.18:

$$\operatorname{Map}_*(S^k, \Omega X) \cong \operatorname{Map}_*(\Sigma S^k, X)$$

□

補題 3.20.

The suspension hom E equals to the composition

↑ 定義 3.13

$$\pi_k(X) \xrightarrow{(\eta_X)_*} \pi_k(\Omega\Sigma X) \xleftarrow[\text{ad}]{\cong} \pi_{k+1}(\Sigma X)$$

↑ 補題 3.19

where

$$\operatorname{Map}_*(\Sigma X, \Sigma X) \xrightarrow[\text{ad}]{\cong} \operatorname{Map}_*(X, \Omega\Sigma X) \quad \text{(↑ 命題 3.18)}$$
$$\mathrm{id}_{\Sigma X} \longmapsto \eta_X$$

証明 (Just a general property for adjunction)

We need to show the commutativity of

$$\begin{array}{ccc} \pi_k(X) & \xrightarrow{E} & \pi_{k+1}(\Sigma X) \\ & \searrow^{(\eta_X)_*} & \downarrow \cong \text{ad} \\ & & \pi_k(\Omega\Sigma X) \end{array}$$

Take $\forall [f] \in \pi_k(X)$ (i.e. $f\colon S^k \to X$)

$\rightsquigarrow$ Enough to show $\mathrm{ad}(\Sigma f) = \eta_X \circ f$

- $\mathrm{ad}(\Sigma f) = (\Omega\Sigma f) \circ \eta_{S^k}$

  $\because$ Naturality of ad:

$$\begin{array}{ccc} \mathrm{id}_{\Sigma S^k} & \longmapsto & \eta_{S^k} \\ \operatorname{Map}_*(\Sigma S^k, \Sigma S^k) & \xrightarrow[\text{ad}]{\cong} & \operatorname{Map}_*(S^k, \Omega\Sigma S^k) \\ \downarrow (\Sigma f)_* & & \downarrow (\Omega\Sigma f)_* \\ \operatorname{Map}_*(\Sigma S^k, \Sigma X) & \xrightarrow[\text{ad}]{\cong} & \operatorname{Map}_*(S^k, \Omega\Sigma X) \\ \Sigma f & \longmapsto & \mathrm{ad}(\Sigma f) = (\Omega\Sigma f) \circ \eta_{S^k} \end{array}$$

- $(\Omega\Sigma f) \circ \eta_{S^k} = \eta_X \circ f$

  $\because$ Naturality of η:

$$\begin{array}{ccc} S^k & \xrightarrow{f} & X \\ \downarrow \eta_{S^k} & & \downarrow \eta_X \\ \Omega\Sigma S^k & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma X \end{array}$$

These complete the proof. □

Now we are ready to prove 定理 3.14.

定理 3.14 の証明 Applying Serre spectral sequence to $\Omega\Sigma X \to P\Sigma X \to \Sigma X$, we have

$$(\eta_X)_*\colon H_k(X) \to H_k(\Omega\Sigma X)\colon \begin{cases} \text{isom} & (k < 2n-1) \\ \text{surj} & (k = 2n-1) \end{cases}$$

$\rightsquigarrow$ $(\eta_X)_*\colon \pi_k(X) \to \pi_k(\Omega\Sigma X)\colon$ (↑ 系 3.10) $\begin{cases} \text{isom} & (k < 2n-1) \\ \text{surj} & (k = 2n-1) \end{cases}$

$\rightsquigarrow$ 補題 3.20 completes the proof □

3.5 Whitehead theorem

定理 3.21 (Whitehead thm)**.**

X, Y: (path-conn) CW complex
$f\colon X \to Y$
↑ space constructed by gluing disks (e.g. mfd)

Assume

$\forall k \in \mathbb{N}$, $f_*\colon \pi_k(X) \xrightarrow{\cong} \pi_k(Y)$: isom

Then

f: homotopy equivalence

(In particular, $\forall k \in \mathbb{N}$, $f_*\colon H_k(X) \xrightarrow{\cong} H_k(Y)$)

$\rightsquigarrow$ $\pi_*(X)$ contains "all info" on homotopy type of X

系 3.22.

X, Y: 1-conn CW complex
↑ = simply conn

$f\colon X \to Y$

Assume

$\forall k \in \mathbb{N}$, $f_*\colon H_k(X;\mathbb{Z}) \xrightarrow{\cong} H_k(Y;\mathbb{Z})$: isom

Then

f: homotopy equivalence

証明 Follows from 定理 3.21 and 系 3.10 □

4 Serre classes

In this section, we

- introduce Serre classes
- prove finiteness of $\pi_k(S^n)$
- explain some properties of torsions of $\pi_k(S^n)$

4.1 Serre classes and Hurewicz isomorphism

定義 4.1 (Serre 1953 [Ser53]).

$\mathcal{C}$: a class of abelian groups

Define

$\mathcal{C}$: Serre class
$\overset{\text{def}}{\iff}$ • $0 \in \mathcal{C}$
• $\forall (0 \to L \to M \to N \to 0)$: exact,
$M \in \mathcal{C} \iff L, N \in \mathcal{C}$

注意 4.2.

(1) 定義自体は [Ser53] によるものだが, (下記の定義 4.3 も含めて) 用語は [Nei10, §2.13] から採用した.
(2) 表現論などで用いられる “Serre subcategory” の語源はおそらくこれ.

定義 4.3 (Serre 1953 [Ser53]).

$\mathcal{C}$: Serre class

Define

- $\mathcal{C}$: Serre ring
  $\overset{\text{def}}{\iff} M, N \in \mathcal{C} \implies M \otimes N,\ \mathrm{Tor}_1^{\mathbb{Z}}(M, N) \in \mathcal{C}$
- $\mathcal{C}$: Serre ideal
  $\overset{\text{def}}{\iff} M \in \mathcal{C} \implies$
  $\forall N$: ab grp, $M \otimes N,\ \mathrm{Tor}_1^{\mathbb{Z}}(M, N) \in \mathcal{C}$
- $\mathcal{C}$: acyclic
  $\overset{\text{def}}{\iff} \forall M \in \mathcal{C}, \forall k > 0, H_k(M; \mathbb{Z}) \in \mathcal{C}$
  ↑ group homology

定義 4.4 (Serre 1953 [Ser53]).

$\mathcal{C}$: Serre class
$f\colon M \to N$: hom of ab grp

Define

- f: $\mathcal{C}$-isom $\overset{\text{def}}{\iff} \mathrm{Ker}\, f,\ \mathrm{Coker}\, f \in \mathcal{C}$
  (i.e. isom “mod $\mathcal{C}$”)
  Denoted by $\cong^{\mathcal{C}}$
- f: $\mathcal{C}$-inj $\overset{\text{def}}{\iff} \mathrm{Ker}\, f \in \mathcal{C}$
- f: $\mathcal{C}$-surj $\overset{\text{def}}{\iff} \mathrm{Coker}\, f \in \mathcal{C}$

例 4.5.

(1) $\mathcal{C} \coloneqq \{0\}$: acyclic Serre ideal
$\rightsquigarrow$ $\mathcal{C}$-isom is the same as the usual isom
(2) $\mathcal{FG} \coloneqq \{\text{fin gen ab grp}\}$: acyclic Serre ring
$\rightsquigarrow$ $M \cong^{\mathcal{FG}} 0 \iff M$: fin gen
(3) $\mathcal{F} \coloneqq \{\text{fin ab grp}\}$: acyclic Serre ring
$\rightsquigarrow$ $M \cong^{\mathcal{F}} 0 \iff M$: fin ab grp
(4) $\mathcal{T} \coloneqq \{\text{torsion ab grp}\}$: acyclic Serre ideal
(5) Let p: prime
$\mathcal{T}_p \coloneqq \{M \in \mathcal{T} \mid M\{p\} = 0\}$: acyclic Serre ideal (定義 4.6)
$\rightsquigarrow$ For $M, N \in \mathcal{T}$,
$M \cong^{\mathcal{T}_p} N \iff M\{p\} \cong N\{p\}$

> If $M, N \notin \mathcal{T}$, we have a counterexample:
> Consider $p\colon \mathbb{Z} \to \mathbb{Z}$ $(M = N = \mathbb{Z})$
> • Since $\mathrm{Coker}(p) \cong \mathbb{Z}/p \notin \mathcal{T}_p$, it is not $\mathcal{T}_p$-isom
> • Obviously $M\{p\} \cong N\{p\}$

定義 4.6.

M: ab. group
p: prime number

Define

$$M\{p\} \coloneqq \{x \in M \mid \exists i \in \mathbb{N}, p^i x = 0\} \subset M$$

定理 4.7 (mod $\mathcal{C}$ Hurewicz isom [Ser53]).

$\mathcal{C}$: acyclic Serre ring
X: 1-connected space
$n > 0$

Assume

$0 < \forall k < n,\ \pi_k(X) \in \mathcal{C}$

Then

(1) $\forall k \le n-1,\ H_k(X; \mathbb{Z}) \in \mathcal{C}$
(2) Hur: $\pi_n(X) \xrightarrow{\cong^{\mathcal{C}}} H_n(X; \mathbb{Z})$: $\mathcal{C}$-isom

証明 By induction on n together with Serre spectral sequence for the fibration $\Omega X \to PX \to X$ □

注意 4.8.

$\mathcal{C}$: acyclic Serre ideal
$\implies$ relative version of mod $\mathcal{C}$ Hurewicz isom

$\because$ (Rough idea)

In the relative case, need to consider

$\Omega X \to (PX, (\mathrm{ev}_1)^{-1}(A)) \xrightarrow{\mathrm{ev}_1} (X, A)$: relative fibration

Here we have

- $H_*(X, A) \in \mathcal{C}$
- (No information on $H_*(\Omega X)$)

Hence

$$H_*(X, A; H_*(\Omega X)) \cong \underbrace{H_*(X, A)}_{\in \mathcal{C}} \otimes H_*(\Omega X) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\cdots) \in \mathcal{C}$$

($\mathcal{C}$: Serre ideal; ignore this for simplicity)

4.2 Finite generation of $\pi_k(S^n)$

定理 4.9 (Serre [Ser51], [Ser53]). (初出 / Serre class で整理・一般化)

X: 1-connected space (See 例 4.11)

Assume

$\forall k \in \mathbb{N}$, $H_k(X; \mathbb{Z})$: fin gen (resp. finite)

Then

$\forall k \in \mathbb{N}$, $\pi_k(X)$: fin gen (resp. finite)

証明 $\mathcal{C} := \{\text{fin gen ab grp}\}$: acyclic Serre ring (resp. finite)

$\rightsquigarrow \forall k,\ H_k(X; \mathbb{Z}) \in \mathcal{C}$ (assump)

$\rightsquigarrow \forall k,\ \pi_k(X) \in \mathcal{C}$ (定理 4.7)

□

系 4.10.

$\pi_k(S^n)$ is finitely generated

証明 $n = 1$ 系 3.2

$n > 1$ 系 4.10

□

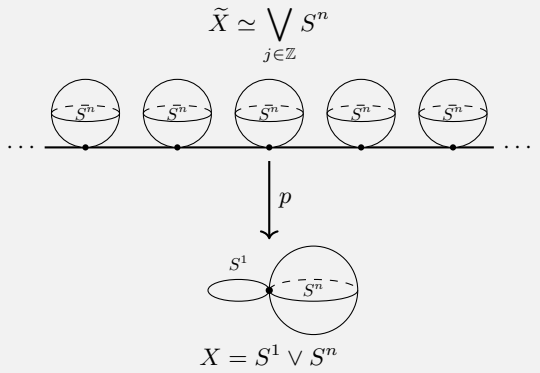
例 4.11.

The assumption of 1-connectivity is necessary for 系 4.10.

For $X = S^1 \vee S^n$ $(n \geq 2)$, we have

- $\forall k$, $H_k(X; \mathbb{Z})$: fin gen
- $\pi_n(X) \cong \pi_n(\widetilde{X}) \cong H_n(\widetilde{X}; \mathbb{Z}) \cong \mathbb{Z}^{\oplus \mathbb{Z}}$: not fin gen

($\widetilde{X}$: universal covering of X; 定理 3.1, 定理 3.7)



定理 4.12 ([Ser51]).

$\pi_k(S^n)$ is finite abelian group except for

- $\pi_n(S^n) \cong \mathbb{Z}$
- $\pi_{4m-1}(S^{2m}) \cong \mathbb{Z} \oplus (\text{fin ab grp})$

証明 Use

- 定理 4.7 for $\mathcal{F} := \{\text{fin ab grp}\}$ (See 例 4.5)
- Cartan-Serre's killing homotopy method (第 6 節)

(See [Hat, Theorem 1.21] for details) □

4.3 Torsions in $\pi_k(S^n)$

定義 4.13.

$n \geq 1$

Define $\eta_n^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$ by the composition

$$\eta_n^2 \colon S^{2n-1} \xrightarrow{\eta_{S^{2n-1}}} \Omega\Sigma S^{2n-1} \cong \Omega S^{2n} \xrightarrow{\Omega\eta_{S^{2n}}} \Omega(\Omega\Sigma S^{2n}) \cong \Omega^2 S^{2n+1}$$

$\rightsquigarrow$

$$\begin{array}{ccc} \pi_k(S^{2n-1}) & \xrightarrow{E^2} & \pi_{k+2}(S^{2n+1}) \\ & \searrow^{(\eta_n^2)_*} & \downarrow \cong \\ & & \pi_k(\Omega^2 S^{2n+1}) \end{array}$$

定理 4.14 ([Ser53]).

$n \geq 1$

p: prime

Then

(1) $E^2 \colon \pi_k(S^{2n-1}) \to \pi_{k+2}(S^{2n+1})$: $\begin{cases} \mathcal{T}_p\text{-isom} & (k < r) \\ \mathcal{T}_p\text{-surj} & (k = r) \end{cases}$

where $r := 2pn - 3$

(2) $\forall k < 2n + 4p - 5$

$$\pi_k(S^{2n+1})\{p\} \cong \pi_{k-2n+2}(S^3)\{p\}$$

証明 **(1)** By Serre spectral sequence, we have

$$(\eta_n^2)_*\colon H_k(S^{2n-1}) \to H_k(\Omega^2 S^{2n+1})\colon \begin{cases} \mathcal{T}_p\text{-isom} & (k<r) \\ \mathcal{T}_p\text{-surj} & (k=r) \end{cases}$$

$\rightsquigarrow$ Apply mod $\mathcal{T}_p$ relative Hurewicz theorem
(c.f. Proof of 定理 3.14)

(2) Apply (1) inductively □

定理 4.15 ([Ser53]).

$$\pi_k(S^3)\{p\} \cong \begin{cases} 0 & (k<2p) \\ \mathbb{Z}/p & (k=2p) \end{cases}$$

証明 Use Cartan-Serre's killing homotopy method
(Detailed proof will be given in 第 6 節) □

4.4 Localization of spaces and exponents of homotopy groups

定理 4.16 ([Tod56], [CMN79], [Nei81], [Nei10, Corollary 11.8.3]).
($p^{2n}\cdot\pi_*=0$ ↑ [Nei10]; ↑ book; ↑ $p\geq 5$; ↑ $p=3$)

p: odd prime
$n \geq 1$
Then

$$p^n \cdot \pi_*(S^{2n+1})\{p\} = 0$$

証明 Throughout this proof, all spaces are localized at p.

- $X \mapsto X_{(p)}$: "localization" of top sp
- $\pi_*(X_{(p)}) \cong \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$
- $H_*(X_{(p)};\mathbb{Z}) \cong H_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

> **where** $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \text{ does not divide } b\} \subset \mathbb{Q}$
> localization of $\mathbb{Z}$ at the prime p

In this proof, we omit $(-)\{p\}$ from the notation
$\rightsquigarrow$ In particular, for $k > 2n+1$,

$$\pi_k(S^{2n+1}) \cong \pi_k(S^{2n+1})\{p\}$$

Here we use the maps

- $\eta_n^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ (定義 4.13)
- $\varphi_n\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ (定理 4.17)

For $k > 2n+1$, we have

$$p\cdot\pi_k(S^{2n+1}) \subset (\eta_n^2)_*(\pi_{k-2}(S^{2n-1}))$$

(↑ identify $\pi_k(S^{2n+1}) \cong \pi_{k-2}(\Omega^2 S^{2n+1})$)

∵

$$\begin{aligned} p\cdot\pi_k(S^{2n+1}) &\cong (\Omega^2 p)_*(\pi_{k-2}(\Omega^2 S^{2n+1})) \\ &\quad (\uparrow = \eta_n^2\circ\varphi_n \text{ by 定理 4.17}) \\ &= (\eta_n^2\circ\varphi_n)_*(\pi_{k-2}(\Omega^2 S^{2n+1})) \\ &\subset (\eta_n^2)_*(\pi_{k-2}(S^{2n-1})) \end{aligned}$$

Hence we have

$$\begin{aligned} p^n\cdot\pi_k(S^{2n+1}) &\subset p^{n-1}\cdot(\eta_n^2)_*(\pi_{k-2}(S^{2n-1})) \\ &\subset p^{n-2}\cdot(\eta_n^2\circ\eta_{n-1}^2)_*(\pi_{k-4}(S^{2n-3})) \\ &\subset \cdots \\ &\subset p^0\cdot(\eta_n^2\circ\cdots\eta_1^2)_*(\pi_{k-2n}(S^1)) \\ &\quad (\uparrow k>2n+1) \\ &= 0 \end{aligned}$$

□

定理 4.17 ([CMN79, Theorem 1.1], [Nei81]).

$\exists\varphi_n\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ s.t.

(1) $\eta_n^2\circ\varphi_n \simeq \Omega^2 p\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$
(2) $\varphi_n\circ\eta_n^2 \simeq p\colon S^{2n-1} \to S^{2n-1}$

> **where**
> $p\colon S^k \to S^k$: the map of degree p
> $(p \in \mathbb{Z} \cong \pi_k(S^k))$

証明 Consider

- $P^{2n+1}(p) \coloneqq S^{2n} \cup_p D^{2n+1}$: mod p Moore space
  $\rightsquigarrow$ $P^{2n+1}(p) \twoheadrightarrow S^{2n+1}$: "pinch map"
- F: its homotopy fiber
  i.e. $F \to P^{2n+1}(p) \to S^{2n+1}$: fibration
  $\rightsquigarrow$ $\Omega^2 S^{2n+1} \xrightarrow{\Omega\partial} \Omega F \to \Omega P^{2n+1}(p)$: fibration

By using

- homotopy Bockstein spectral sequence
- Samelson product (Lie alg)

we have a decomposition

$$\Omega F \simeq S^{2n-1} \times (\text{other factors})$$

Combining this with $\Omega\partial$, we obtain

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega\partial} \Omega F \xrightarrow{\text{proj}} S^{2n-1}$$

□

Moreover, 定理 4.16 is best-possible by the following theorem:

定理 4.18 ([Gra69], [Bau81, Theorem 1.9, pp. 93–94]).

lecture note

p: odd prime

$k > n \geq 1$ with $p^{n-1} \mid k$

Then

$$p^{n-1} \cdot \pi_{2k(p-1)+2n}(S^{2n+1})\{p\} \neq 0$$

5 Fibration

参考文献

[Hat02] §4.2 内の “Fiber bundles” (p. 375〜) や §4.3 内の “Fibrations” (p. 405〜) が参考になる

[玉木 20] 第 5 章が丸ごと fibration に関する章で，非常に詳しく書かれている．

5.1 Fibration and homotopy groups

定義 5.1.

$p\colon E \to B$: map (B: path-connected)

Define

p: (Hurewicz) fibration ← resp. Serre fibration

$\overset{\text{def}}{\iff} \forall Z$: space ← resp. $Z = D^n$ for $n \in \mathbb{N}$

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\forall} & E \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow p \\ Z \times I & \xrightarrow{\forall} & B \end{array}$$

$\rightsquigarrow$

- Take $b_0 \in B$ (base point)
- $F \coloneqq p^{-1}(b_0)$: the fiber of p
  ↑ homotopy type of F is indep of b_0
- For short, we write
  $F \to E \xrightarrow{p} B$: fibration

例 5.2.

- covering space
- fiber bundle (with paracompact base)
  $\rightsquigarrow$ $E =$ (“twisted product” of F and B)
- $\Omega X \to PX \xrightarrow{\mathrm{ev}_1} X$: path-loop fibration

  where
  - (X, x_0): based space
  - $PX \coloneqq \{\gamma\colon I \to X \mid \gamma(0) = x_0\}$: based path space
  - $\Omega X \coloneqq \mathrm{Map}_*(S^1, X)$ based loop space
    $\cong \{\gamma\colon I \to X \mid \gamma(0) = \gamma(1) = x_0\}$
    $\subset PX$
    (see 定義 3.17)
  - $\mathrm{ev}_1\colon PX \longrightarrow X$, $\gamma \longmapsto \gamma(1)$

  (Apply 命題 5.12 to $\{x_0\} \hookrightarrow X$)

注意 5.3.

We can regard $F \to E \to B$ as a “short exact seq of spaces”

$\because$

- $F = p^{-1}(b_0) =$ “$\mathrm{Ker}\, p$”
- $E =$ (“twisted product” of F and B)

$\rightsquigarrow$ Are there long exact sequences?

- For homotopy groups: Yes! (命題 5.4)
- For (co)homology groups: Partially yes! (第 5.2 節)

命題 5.4.

$F \xrightarrow{i} E \xrightarrow{p} B$: fibration

Then The following seq is exact:

$$\begin{aligned} \cdots \longrightarrow \pi_k(F) &\xrightarrow{i_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B) \\ \longrightarrow \pi_{k-1}(F) &\xrightarrow{i_*} \pi_{k-1}(E) \xrightarrow{p_*} \pi_{k-1}(B) \\ \longrightarrow \pi_{k-2}(F) &\xrightarrow{i_*} \cdots \end{aligned}$$

(Around π_0: exact seq of based sets)

証明

- The long exact seq for the pair (E, F):

$$\begin{aligned} \cdots \longrightarrow \pi_k(F) &\xrightarrow{i_*} \pi_k(E) \longrightarrow \pi_k(E, F) \\ \longrightarrow \pi_{k-1}(F) &\xrightarrow{i_*} \cdots \end{aligned}$$

- $p_*\colon \pi_k(E, F) \xrightarrow{\cong} \pi_k(B)$

(See [Hat02, Theorem 4.41]) □

系 5.5.

$$\pi_3(S^2) \cong \mathbb{Z}$$

証明 Recall: $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = S^{2n+1}/S^1$
(Restrict to the unit sphere ↑)

$\rightsquigarrow$ $S^1 \to S^{2n+1} \to \mathbb{C}P^n$: fibration

For $n = 1$, we have

- $S^1 \to S^3 \to \mathbb{C}P^1$: fibration
- $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \cong S^2$

$\rightsquigarrow$ $S^1 \to S^3 \to S^2$: fibration
called the Hopf fibration
(See [Hat02, Example 4.45], [Wikd])

By 命題 5.4, we have

$$\cdots \longrightarrow \underbrace{\pi_3(S^1)}_{=0} \longrightarrow \underbrace{\pi_3(S^3)}_{\cong \mathbb{Z}} \longrightarrow \pi_3(S^2) \longrightarrow \underbrace{\pi_2(S^1)}_{=0} \longrightarrow \pi_2(S^3) \longrightarrow \cdots$$

$\rightsquigarrow$ $\mathbb{Z} \cong \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$ □

5.2 Fibration and cohomology groups

Here we discuss "long exact seq" of cohomology for a fibration. (See 注意 5.3)

注意 5.6.

In contrast to the case of homotopy groups, we obtain:
(complicated version of exact seq)

- General case $\rightsquigarrow$ Serre spectral sequence (定理 5.7)
- $B \simeq S^n$ $\rightsquigarrow$ Wang exact seq (定理 5.8)
- $F \simeq S^n$ $\rightsquigarrow$ Gysin exact seq (定理 5.10)

定理 5.7 ([Ser51]).

$F \to E \to B$: fibration

G: ab grp

Then $\exists \{E_r^{p,q}\}$: 1st quadrant spectral seq s.t.

- $E_2^{p,q} \cong H^p(B; H^q(F;G))$
  (local coeff (if $\pi_1(B) \neq 0$))
- "converges to" $H^{p+q}(E;G)$
- $1 \leq \forall r \leq \infty$
  - E_r is a bigraded algebra
  - $d_r : E_r \to E_r$: derivation

定理 5.8 ([Wan49], [Ser51]).
(幾何的証明) (Serre スペクトル系列を用いて一般化)

$n \geq 2$

$F \xrightarrow{i} E \xrightarrow{p} B$: fibration

Assume

- B: 1-connected
- $H^*(B;\mathbb{Z}) \cong H^*(S^n;\mathbb{Z})$

Then The following seq is exact:

$$\begin{aligned} \cdots \longrightarrow H^k(E) &\xrightarrow{i^*} H^k(F) \xrightarrow{\theta} H^{k-n+1}(F) \\ &\longrightarrow H^{k+1}(E) \xrightarrow{i^*} H^{k+1}(F) \xrightarrow{\theta} H^{k-n+2}(F) \\ &\longrightarrow H^{k+2}(E) \xrightarrow{i^*} \cdots \end{aligned}$$

Moreover, θ is a derivation of degree $-(n-1)$
i.e. for $\forall x, y \in H^*(F)$,

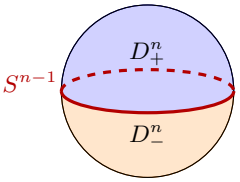
$$\theta(xy) = \theta(x)y + (-1)^{|x|(n-1)} x\theta(y)$$

where $|x|$: degree of x (i.e. $x \in H^{|x|}(F)$)

証明 (See [Ser51] [服部 91, §10.4 (b)] for details)
Serre スペクトル系列の Corollary として直ちに得られる □

別証明 (See [Die08, Theorem 17.7.2] [小中菅 67, 第 6 章 定理 6.1] for details)
For simplicity, we assume $B = S^n$.

- Write $S^n = D_+^n \cup D_-^n$
  Identify $S^{n-1} = D_+^n \cap D_-^n \subset S^n$
- Define
  - $E_\pm := p^{-1}(D_\pm^n)$
  - $E_0 := E_+ \cap E_- = p^{-1}(S^{n-1})$



$S^n = D_+^n \cup D_-^n$

$\rightsquigarrow$ We have

- $E_\pm \cong D^n \times F \simeq F$ ($\because D_\pm^n$ is contractible)
  $\rightsquigarrow$ $H^k(E_\pm) \cong H^k(F)$
- $E_0 \cong S^{n-1} \times F$ ($\because$ restriction of $E_+ \cong D^n \times F$)
  (We choose +)
  $\rightsquigarrow$ $H^k(E_0) \cong H^k(F) \oplus H^{k-n+1}(F)$

By Mayer-Vietoris seq, we have

$$\cdots \longrightarrow H^k(E) \longrightarrow \underbrace{H^k(E_+)}_{\cong H^k(F)} \oplus \underbrace{H^k(E_-)}_{\cong H^k(F)} \xrightarrow{\tilde{\theta}} \underbrace{H^k(E_0)}_{H^k(F) \oplus H^{k-n+1}(F) \cong} \longrightarrow H^{k+1}(E) \longrightarrow \cdots$$

Here we can write

$$\tilde{\theta} = \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \theta \end{pmatrix} : H^k(F) \oplus H^k(F) \to H^k(F) \oplus H^{k-n+1}(F)$$

($\because$ We chose + to trivialize E_0)

where

- $\alpha : S^{n-1} \times F \to F$: the "clutching function" ([Wika])

  α determines the gluing

  $$(D_+^n \times F) \cup_{S^{n-1} \times F} (D_-^n \times F) = E_+ \cup E_- = E$$

- By 補題 5.9, we have $\theta := \theta_\alpha : H^k(F) \to H^{k-n+1}(F)$
  $\rightsquigarrow$ θ: derivation of deg $-(n-1)$

"Canceling" the part id: $H^k(F) \xrightarrow{\cong} H^k(F)$,
we obtained the seq in Theorem. □

補題 5.9.

$m \geq 1$ ($m = n-1$ in 定理 5.8)

$b_0 \in S^m$

F: top sp

$\alpha : S^m \times F \to F$ s.t. $\alpha|_{\{b_0\} \times F} \simeq \mathrm{id}_F : F \to F$

$\rightsquigarrow$ We have

$$\alpha^* : H^k(F) \to H^k(S^m \times F)$$

$$\cong (H^0(S^m) \otimes H^k(F)) \oplus (H^m(S^m) \otimes H^{k-m}(F))$$

(Künneth thm [Hat02, Thm 3.16])

$\rightsquigarrow$ Define $\theta_\alpha : H^*(F) \to H^{*-m}(F)$ by

$$\alpha^*(x) = 1 \otimes x + \omega \otimes \theta_\alpha(x)$$

(assump: $\alpha|_{\{b_0\} \times F} \simeq \mathrm{id}_F$)

where

- $1 \in H^0(S^m)$: the unit
- $\omega \in H^m(S^m)$: generator

Then θ_α is a derivation of deg $-m$
Namely, for $\forall x, y \in H^*(F)$,

$$\theta_\alpha(xy) = \theta_\alpha(x) \cdot y + (-1)^{m|x|} x \cdot \theta_\alpha(y)$$

証明 Write $\theta := \theta_\alpha$ for short.
Since α^* is an alg map, we have

$$\alpha^*(xy) = \alpha^*(x)\alpha^*(y)$$

Here we can write these by using θ:

- $\alpha^*(xy) = 1 \otimes xy + \omega \otimes \theta(xy)$
- $\alpha^*(x)\alpha^*(y) = (1 \otimes x + \omega \otimes \theta(x))(1 \otimes y + \omega \otimes \theta(y))$
  $= 1 \otimes xy + \omega \otimes (\theta(x)y + (-1)^{|x||\omega|} x\theta(y))$ □

定理 5.10 ([Gys41], [Ser51]).

$n \geq 1$
$F \to E \xrightarrow{p} B$: fibration
Assume

- B: 1-connected
- $H^*(F;\mathbb{Z}) \cong H^*(S^n;\mathbb{Z})$

Then The following seq is exact:

$$\begin{aligned} \cdots \longrightarrow H^{k-n-1}(B) &\xrightarrow{\cup e} H^k(B) \xrightarrow{p^*} H^k(E) \\ &\xrightarrow{p_!} H^{k-n}(B) \xrightarrow{\cup e} H^{k+1}(B) \xrightarrow{p^*} H^{k+1}(E) \\ &\xrightarrow{p_!} H^{k-n+1}(B) \xrightarrow{\cup e} \cdots \end{aligned}$$

where

- $e \in H^{n+1}(B;\mathbb{Z})$: the Euler class
- $p_!$: integration along the fiber

証明 Follows from either of

- Serre spectral sequence 定理 5.7
- Thom isomorphism theorem

□

5.3 Homotopy fiber

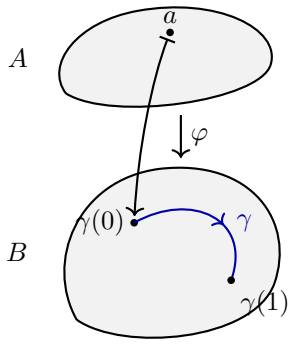
定義 5.11.

$\varphi: A \to B$: continuous map
$b_0 \in B$
Define

- $E_\varphi := \{(a,\gamma) \mid \gamma(0) = \varphi(a)\} \subset A \times B^I$
  $p_\varphi: E_\varphi \longrightarrow B,\ (a,\gamma) \longmapsto \gamma(1)$
- $F_\varphi := \{(a,\gamma) \mid \gamma(0) = \varphi(a),\ \gamma(1) = b_0\} \subset E_\varphi$
  $i_\varphi: F_\varphi \hookrightarrow E_\varphi$: the inclusion

We call
F_φ: homotopy fiber

These are closely related to cylinder and cone (第 5.4 節)



$(a,\gamma) \in E_\varphi$

$$\begin{array}{ccc} E_\varphi & \longrightarrow & B^I \\ \simeq\downarrow & & \simeq\downarrow \mathrm{ev}_0 \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

命題 5.12.

(1) $A \xleftarrow{\simeq} E_\varphi$ homotopy equiv
$a \longmapsto (a, \mathrm{const}_{\varphi(a)})$
$a \longleftarrow\!\shortmid (a,\gamma)$ ↖ const path at $\varphi(a)$

(2) $F_\varphi \xrightarrow{i_\varphi} E_\varphi \xrightarrow{p_\varphi} B$: fibration
(Sometimes we write $F_\varphi \to A \to B$: fibration)

証明

(1) For $\forall (a,\gamma) \in E_\varphi$, no condition is assumed for $\gamma(1)$
⇝ We can shorten γ to $\mathrm{const}_{\varphi(a)}$ (in E_φ)
⇝ We obtain a deformation retract to

$$\underbrace{\{(a, \mathrm{const}_{\varphi(a)}) \mid a \in A\}}_{\cong A} \subset E_\varphi$$

(2) Given a diagram

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\forall h} & E_\varphi \\ \downarrow & \overset{\exists ?}{\nearrow} & \downarrow p_\varphi \\ Z \times I & \xrightarrow{\forall H} & B \end{array}$$

⇝ Write $h(Z) = (a(Z), \gamma_Z)$ $(Z \in Z)$

where

- $a: Z \to A$
- $\gamma_Z: I \to B$ (depends continuously on $Z \in Z$)

such that

- $\gamma_Z(0) = \varphi(a(Z))$
- $\gamma_Z(1) = H_Z(0)$ $\quad (H_Z(t) := H(Z,t))$

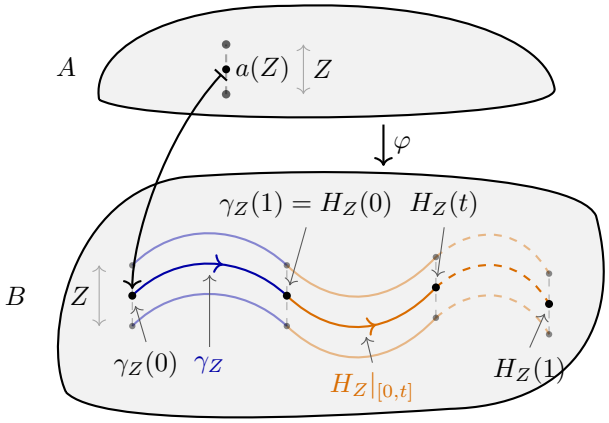
For $Z \in Z$ and $t \in I$, define a path $\Gamma_{Z,t}: I \to B$ by

$$\Gamma_{Z,t}(u) := \begin{cases} \gamma_Z\left(\dfrac{u}{1-t/2}\right) & (0 \leq u \leq 1 - t/2), \\ H_Z(2u - 2 + t) & (1 - t/2 \leq u \leq 1). \end{cases}$$

Using this, we define $\tilde{H}: Z \times I \to E_\varphi$ by

$$\tilde{H}(Z,t) := (a(Z), \Gamma_{Z,t})$$

$\rightsquigarrow \begin{cases} \tilde{H}(Z,0) = h(Z) \\ p_\varphi(\tilde{H}(Z,t)) = H(Z,t) \end{cases}$ as required



$h(Z) = (a(Z), \gamma_Z), \quad \tilde{H}(Z,t) = (a(Z), \Gamma_{Z,t}) \quad (Z \in Z)$

(See [Hat02, Proposition 4.64] for details) □

命題 5.13.

$\varphi\colon (A, a_0) \to (B, b_0)$: based continuous map

$\rightsquigarrow F_\varphi \xrightarrow{i_\varphi} E_\varphi \xrightarrow{p_\varphi} B$: fibration

$\rightsquigarrow F_{i_\varphi} \xrightarrow{i_{i_\varphi}} E_{i_\varphi} \xrightarrow{p_{i_\varphi}} E_\varphi$: fibration

Then

$$F_{i_\varphi} \simeq \Omega B$$

$\rightsquigarrow \Omega B \to F_\varphi \to A \to B$: "homotopy fiber seq"

証明 We have a comm diagram

$$\begin{array}{ccc} F_\varphi & \xrightarrow{i_\varphi} & E_\varphi \\ & \searrow^{q} & \downarrow{\scriptstyle \simeq}\, \eta \\ & & A \end{array}$$

where

- η: $\begin{array}{ccc} E_\varphi & \xrightarrow{\simeq} & A \\ (a,\gamma) & \longmapsto & a \end{array}$: homotopy equiv (命題 5.12 (1))
- $q \coloneqq \eta \circ i_\varphi$: $\begin{array}{ccccc} F_\varphi & \xrightarrow{i_\varphi} & E_\varphi & \xrightarrow[\simeq]{\eta} & A \\ (a,\gamma) & & \longmapsto & & a \end{array}$

$\rightsquigarrow F_{i_\varphi} \simeq F_q$

$\because$ homotopy fiber "preserves" homotopy equiv

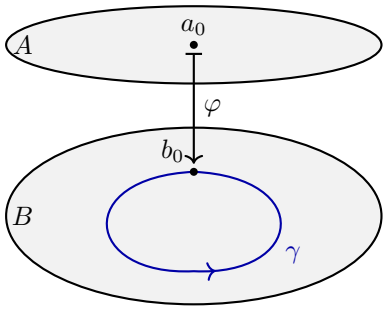
Moreover,

$q\colon F_\varphi \to A$: fibration

$\rightsquigarrow F_q \simeq q^{-1}(a_0) \cong \Omega B$

$\because$

$$\begin{aligned} q^{-1}(a_0) &= \{(a,\gamma) \in F_\varphi \mid a = a_0\} \\ &= \{(a,\gamma) \in A \times B^I \mid \\ &\qquad \underbrace{\varphi(a) = \gamma(0), \gamma(1) = b_0}_{(a,\gamma) \in F_\varphi}, \quad \underbrace{a = a_0}_{q((a,\gamma)) = a_0}\} \\ &\cong \{\gamma \in B^I \mid \varphi(a_0) = \gamma(0), \gamma(1) = b_0\} \\ &\cong \Omega B \end{aligned}$$



□

系 5.14.

$\varphi\colon A \to B$: continuous map

$b_0 \in B$

Then By taking successive homotopy fibers, we have

$$\begin{aligned} \cdots \longrightarrow \Omega^2 F_\varphi &\xrightarrow{\Omega^2 i} \Omega^2 A \xrightarrow{\Omega^2 \varphi} \Omega^2 B \\ &\longrightarrow \Omega F_\varphi \xrightarrow{-\Omega i} \Omega A \xrightarrow{-\Omega \varphi} \Omega B \\ &\longrightarrow F_\varphi \xrightarrow{i} A \xrightarrow{\varphi} B \end{aligned}$$

s.t. each successive triple is a fibration (up to homotopy)

証明 Follows from

- 命題 5.13
- homotopy fiber "preserves" homotopy equiv

□

注意 5.15.

(1) The seq in 系 5.14 is a topological version of "exact triangle" (in triangulated category).

(2) Given $F \to E \xrightarrow{p} B$: fibration

$\rightsquigarrow F_p \simeq F$ (See [Hat02, Prop 4.65])

Applying the functor π_k to the seq, we have

$$\begin{aligned} \cdots \longrightarrow \underbrace{\pi_k(\Omega^2 F)}_{\cong \pi_{k+2}(F)} &\longrightarrow \underbrace{\pi_k(\Omega^2 E)}_{\cong \pi_{k+2}(E)} \longrightarrow \underbrace{\pi_k(\Omega^2 B)}_{\cong \pi_{k+2}(B)} \\ \longrightarrow \underbrace{\pi_k(\Omega F)}_{\cong \pi_{k+1}(F)} &\longrightarrow \underbrace{\pi_k(\Omega E)}_{\cong \pi_{k+1}(E)} \longrightarrow \underbrace{\pi_k(\Omega B)}_{\cong \pi_{k+1}(B)} \\ \longrightarrow \pi_k(F) &\longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \end{aligned}$$

$\rightsquigarrow$ We recover the exact seq in 命題 5.4.

5.4 Mapping cone and cylinder

(この subsection の内容は、以降では使わない)

5.4.1 For spaces

There are "dual" notions for fibration and homotopy fiber.
Eckmann–Hilton duality [Wikb]

See [玉木 20, §5.9, 5.10] for details.

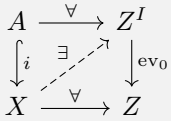
定義 5.16.

$i\colon A \to X$: map

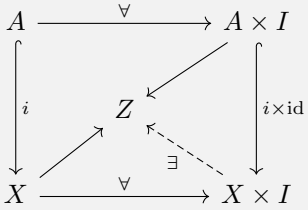
Define

i: cofibration

$\overset{\text{def}}{\iff} \forall Z$: space



$\iff \forall Z$: space



- $X/A \coloneqq X/\sim$: the cofiber of i

  where $\forall a, a' \in A, \quad a \sim a'$

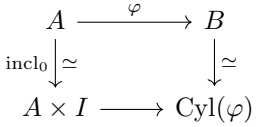
- We also write
  $A \xrightarrow{i} X \to X/A$: cofibration

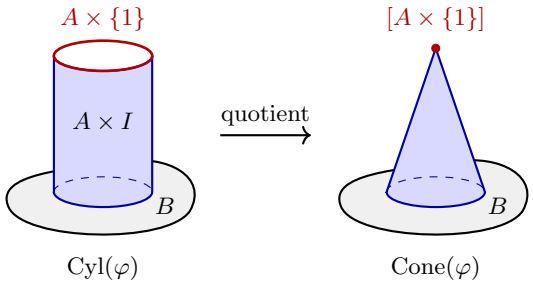
定義 5.17.

$\varphi\colon A \to B$: conti map

Define

- $\mathrm{Cyl}(\varphi) \coloneqq (A \times I) \cup_A B$: mapping cylinder
- $\mathrm{Cone}(\varphi) \coloneqq \mathrm{Cyl}(\varphi)/(A \times \{1\})$: mapping cone





補題 5.18.

(1) $B \simeq \mathrm{Cyl}(\varphi)$

(2) $A \to \mathrm{Cyl}(\varphi) \to \mathrm{Cone}(\varphi)$: cofibration

(3) $A \to B \to \mathrm{Cone}(\varphi) \to \Sigma A$: "htpy cofiber seq"

5.4.2 For chain complexes

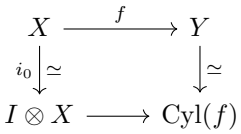
There are algebraic counterparts ([Wei94, §1.5]):

定義 5.19.

$f\colon X \to Y$: chain map ($d\colon X_k \to X_{k-1}$)

Define

- $\mathrm{Cyl}(f)$: chain complex given by
  - $\mathrm{Cyl}(f)_k \coloneqq X_k \oplus X_{k-1} \oplus Y_k$
  - $d_k \coloneqq \begin{pmatrix} d_X & \mathrm{id}_X & 0 \\ 0 & -d_X & 0 \\ 0 & -f & d_Y \end{pmatrix} : \mathrm{Cyl}(f)_k \to \mathrm{Cyl}(f)_{k-1}$
- $\mathrm{Cone}(f)$: chain complex given by
  - $\mathrm{Cone}(f)_k \coloneqq X_{k-1} \oplus Y_k$
  - $d_k \coloneqq \begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix} : \mathrm{Cone}(f)_k \to \mathrm{Cone}(f)_{k-1}$



where

- $I \coloneqq (\cdots \to 0 \to \mathbb{Z}e \xrightarrow{\partial} \mathbb{Z}v_0 \oplus \mathbb{Z}v_1 \to 0 \to \cdots)$
  deg 1, deg 0
- $\partial(e) \coloneqq v_1 - v_0$
- $i_0(x) \coloneqq v_0 \otimes x$

補題 5.20.

(1) $\mathrm{Cyl}(f) \simeq Y$

(2) $0 \to X \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to 0$: exact

(3) $X \to Y \to \mathrm{Cone}(f) \to X[1]$: "exact triangle"
shift of X, i.e. $X[1]_n = X_{n-1}$

(4) $Y[-1] \to \mathrm{Cone}(f)[-1] \to X \to Y$: "exact triangle"

5.4.3 Difference between spaces and chain complexes

注意 5.21.

spaces are unstable

6 Killing homotopy の方法

[CS52a] [CS52b] にて与えられた，いわゆる “killing homotopy” の方法について解説する．

Idea

X: top. space

$n \in N$

$\rightsquigarrow$ We want to compute $\pi_n(X)$

(1) Construct a space X' such that
- $\forall k < n,\ \pi_k(X') = 0$
- $\pi_n(X') \cong \pi_n(X)$

(2) Compute $H_*(X'; \mathbb{Z})$

(3) We have $H_n(X;\mathbb{Z}) \cong \pi_n(X') \cong \pi_n(X)$ (定理 3.7 (Hurewicz thm))

Keys to construct such a space X':
- Eilenberg-MacLane space $K(G,n)$ (定理 6.1)
- homotopy fiber (定義 5.11)

6.1 Eilenberg-MacLane space

定理 6.1.

G: abelian group ($n = 1$ なら non-abel でも OK)

$n \geq 1$

Then $\exists! K(G,n)$: space (unique up to homotopy equiv) s.t.

$$\pi_k(K(G,n)) \cong \begin{cases} G & (k = n) \\ 0 & (k \neq n) \end{cases}$$

$K(G,n)$ is called Eilenberg-MacLane space of type (G,n)

証明

Existence Take a free resol. as an ab. group:

$$0 \to \mathbb{Z}B \xrightarrow{f} \mathbb{Z}A \to G \to 0$$

where
- A, B: (possibly infinite) sets
- $\mathbb{Z}A$: free ab. group with basis A
- $\mathbb{Z}B$: free ab. group with basis B

By induction on $m \geq n$, we construct K_m s.t.
- K_m: CW cpx of dim $\leq m$
- $K_{m-1} \subset K_m$
- $\forall k < m,\ \pi_k(K_m) \cong \pi_k(K(G,n))$

$\rightsquigarrow$ Define $K(G,n) \coloneqq \bigcup_m K_m$

$m = n$ Define $K_n \coloneqq \bigvee_A S^n$

$\rightsquigarrow \pi_k(K_n) \cong \begin{cases} \mathbb{Z}A & (k = n) \\ 0 & (k < n) \end{cases}$ (定理 3.7 (Hurewicz thm))

$m = n+1$ Define $\varphi_b \in \pi_n(K_n)$ by

$$\begin{array}{ccc} \pi_n(K_n) & \xleftarrow{\cong} & \mathbb{Z}A \\ \varphi_b & \longmapsto & f(b) \end{array}$$

(see the free resol)

$\rightsquigarrow$ Attach $(n+1)$-cells along φ_b for each $b \in B$:

$$K_{n+1} \coloneqq K_n \cup \bigcup_B D^{n+1}$$

In other words, define K_{n+1} by the pushout diagram

$$\begin{array}{ccc} \coprod_{b\in B} S^n & \xrightarrow{\coprod_b \varphi_b} & K_n \\ {\scriptstyle \coprod_b \mathrm{incl}}\downarrow & & \downarrow \\ \coprod_{b\in B} D^{n+1} & \longrightarrow & K_{n+1} \end{array}$$

$\rightsquigarrow$ Each $\varphi_b \in \pi_n(K_n)$ is killed by D^{n+1} corresponding to $b \in B$

$\rightsquigarrow \pi_n(K_n) \cong \mathbb{Z}A / f(\mathbb{Z}B) \cong G$

$m > n+1$ Attach m-cells to kill all elements of $\pi_{m-1}(K_{m-1})$ without affecting $\pi_{<m-1}(K_{m-1})$

Uniqueness Omit

(See [Hat02, Proposition 4.30] for details) □

例 6.2.
- $K(\mathbb{Z},1) \simeq S^1$
- $K(\mathbb{Z}/2,1) \simeq \mathbb{R}P^\infty$
- $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$

補題 6.3.

$$\Omega K(G,n) \simeq K(G,n-1)$$

証明

$$\pi_k(\Omega K(G,n)) \cong \pi_{k+1}(K(G,n)) \cong \begin{cases} G & (k+1 = n) \\ 0 & (k+1 \neq n) \end{cases}$$

(補題 3.19) □

命題 6.4.

$$H^*(K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z}[x]$$

(poly ring with $|x| = 2$)

証明 Use Gysin seq 定理 5.10 □

6.2 Cartan-Serre's killing homotopy method

定理 6.5.

$$\pi_4(S^3) \cong \mathbb{Z}/2$$

証明 Take $[\varphi] \in \pi_3(K(\mathbb{Z},3)) \cong \mathbb{Z}$: generator (by def of $K(\mathbb{Z},3)$)

$\rightsquigarrow$
- $\varphi\colon S^3 \to K(\mathbb{Z},3)$
- $\varphi_*\colon \pi_3(S^3) \xrightarrow{\cong} \pi_3(K(\mathbb{Z},3))$: isom

$\rightsquigarrow$ Take the homotopy fiber of φ:

$$F_\varphi \to S^3 \to K(\mathbb{Z},3)$$

$\rightsquigarrow$ We have

$$\pi_k(F_\varphi) \cong \begin{cases} \pi_k(S^3) & (k \geq 4) \\ 0 & (k \leq 3) \end{cases}$$

$\because$

$$\cdots \longrightarrow \underbrace{\pi_{k+1}(K(\mathbb{Z},3))}_{=0}$$

$$\longrightarrow \pi_k(F_\varphi) \xrightarrow{\cong} \pi_k(S^3) \longrightarrow \underbrace{\pi_k(K(\mathbb{Z},3))}_{=0} \quad (k \geq 4)$$

$$\longrightarrow \pi_{k-1}(F_\varphi) \xrightarrow{\cong} \cdots$$

$$\cdots \longrightarrow \underbrace{\pi_4(K(\mathbb{Z},3))}_{=0}$$

$$\longrightarrow \underbrace{\pi_3(F_\varphi)}_{=0} \longrightarrow \pi_3(S^3) \xrightarrow{\cong} \pi_3(K(\mathbb{Z},3))$$

$$\longrightarrow \underbrace{\pi_2(F_\varphi)}_{=0} \longrightarrow \underbrace{\pi_2(S^3)}_{=0} \longrightarrow \underbrace{\pi_2(K(\mathbb{Z},3))}_{=0}$$

$$\longrightarrow \underbrace{\pi_1(F_\varphi)}_{=0} \longrightarrow \underbrace{\pi_1(S^3)}_{=0} \longrightarrow \pi_1(K(\mathbb{Z},3))$$

$\rightsquigarrow$ $\pi_4(S^3) \cong \pi_4(F_\varphi) \cong H_4(F_\varphi;\mathbb{Z})$ (定理 3.7)

$\rightsquigarrow$ Hence 補題 6.10 below completes the proof □

補題 6.6.

$K(\mathbb{Z},2) \to F_\varphi \to S^3$: fibration

証明 Taking the homotopy fiber for $F_\varphi \to S^3$, 命題 5.13 completes the proof □

補題 6.7.

$$H^k(F_\varphi;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k=0) \\ \mathbb{Z}/m & (k = 2m+1 \geq 5) \\ 0 & (\text{otherwise}) \end{cases}$$

証明 Write $K := K(\mathbb{Z},2)$

$\rightsquigarrow$ $H^*(K) \cong \mathbb{Z}[x]$ (命題 6.4; x: deg 2)

By 補題 6.6, we have

$$K \to F_\varphi \to S^3\text{: fibration}$$

Here we have

$$0 < \forall k \leq 3, \quad H^k(F_\varphi) = 0 \tag{6.8}$$

$\because$
- $\pi_{\leq 3}(F_\varphi) = 0$
- Hurewicz thm (定理 3.7)
- universal coefficient theorem (see 補題 6.10 for ref)

Applying the Wang exact seq (定理 5.8) to the fibration in 補題 6.6, we have

$$\cdots \longrightarrow \underbrace{H^2(F_\varphi)}_{=0\ (\because (6.8))} \longrightarrow H^2(K) \xrightarrow[\cong]{\theta} H^0(K)$$

$$\longrightarrow \underbrace{H^3(F_\varphi)}_{=0\ (\because (6.8))} \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{2m-1}(K) \xrightarrow{\theta} \underbrace{H^{2m-3}(K)}_{=0}$$

$$\longrightarrow H^{2m}(F_\varphi) \longrightarrow H^{2m}(K) \xrightarrow{\theta} H^{2m-2}(K)$$

$$\longrightarrow H^{2m+1}(F_\varphi) \longrightarrow \underbrace{H^{2m+1}(K)}_{=0} \xrightarrow{\theta} \cdots$$

$\rightsquigarrow$ We have an exact seq

$$0 \to H^{2m}(F_\varphi) \to H^{2m}(K) \xrightarrow{\theta} H^{2m-2}(K) \to H^{2m+1}(F_\varphi) \to 0 \tag{6.9}$$

$\rightsquigarrow$ Enough to compute $\theta\colon H^{2m}(K) \to H^{2m-2}(K)$ (for $m \geq 2$)

Here we have

- $\theta\colon \underbrace{H^2(K)}_{\cong \mathbb{Z}x} \longrightarrow \underbrace{H^0(K)}_{\cong \mathbb{Z}1}$, $x \longmapsto \pm 1$

  $\rightsquigarrow$ We may assume $\theta(x) = 1$

- $\theta\colon \underbrace{H^{2m}(K)}_{\cong \mathbb{Z}x^m} \longrightarrow \underbrace{H^{2(m-1)}(K)}_{\cong \mathbb{Z}x^{m-1}}$, $x^m \longmapsto m x^{m-1}$

  $\because$ θ: derivation

  $\rightsquigarrow$

$$\begin{aligned} \theta(x^m) &= \theta(x \cdot x^{m-1}) \\ &= \underbrace{\theta(x)}_{=1} \cdot x^{m-1} + (-1)^{2\cdot 2} x \cdot \underbrace{\theta(x^{m-1})}_{=(m-1)x^{m-2}\ \text{by ind}} \\ &= m x^{m-1} \end{aligned}$$

$\rightsquigarrow$ (6.9) can be written as

$$0 \to H^{2m}(F_\varphi) \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to H^{2m+1}(F_\varphi) \to 0$$

$\rightsquigarrow$
- $H^{2m}(F_\varphi) = 0$
- $H^{2m+1}(F_\varphi) \cong \mathbb{Z}/m$

(See [Hat, Example 5.19] for details) □

補題 6.10.

$$H_k(F_\varphi;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k=0) \\ \mathbb{Z}/m & (k=2m\geq 4) \\ 0 & (\text{otherwise}) \end{cases}$$

証明 補題 6.7 and the universal coefficient theorem [Hat02, Theorem 3.2 and p.198] □

補題 6.10 completes the proof of 系 5.5.

stated in 第 4 節

Moreover, we can prove 定理 4.15.

Recall from 定義 4.6:

$$M\{p\} \coloneqq \{x \in M \mid \exists i \in \mathbb{N}, p^i x = 0\} \subset M$$

定理 4.15 ([Ser53]).

$$\pi_k(S^3)\{p\} \cong \begin{cases} 0 & (k<2p) \\ \mathbb{Z}/p & (k=2p) \end{cases}$$

証明 By 補題 6.10,

$$H_k(F_\varphi;\mathbb{Z}) \cong^{\mathcal{T}_p} \begin{cases} 0 & (k<2p) \\ \mathbb{Z}/p & (k=2p) \end{cases}$$

where Recall from 例 4.5:
- $\mathcal{T}_p \coloneqq \{M \in \mathcal{T} \mid M\{p\} = 0\}$
- For $M, N \in \mathcal{T}$,
  $M \cong^{\mathcal{T}_p} N \iff M\{p\} \cong N\{p\}$

$\rightsquigarrow$ By 定理 4.7 (mod $\mathcal{C}$ Hurewicz isom), we have

$$\pi_k(F_\varphi) \cong^{\mathcal{T}_p} \begin{cases} 0 & (k<2p) \\ \mathbb{Z}/p & (k=2p) \end{cases}$$

Since $\forall k \geq 4$, $\pi_k(S^3) \cong \pi_k(F_\varphi)$, this completes the proof. □

7 Rational homotopy theory

この節はまだ全然書けていません

- Sullivan model の説明
- minimal model の生成元とホモトピー群の同型
- 球面の minimal model
- $\mathbb{C}P^2 \# \mathbb{C}P^2$ (向き反転？ 3 つ以上？)
- $S^n \vee S^n$
- model of pullback (free loop space)

Notation

Throughout this section, coefficients are taken in $\mathbb{Q}$ (or a field of char 0)

7.1 Sullivan model

定義 7.1.

- DGA
- (graded) commutative

定義 7.2.

$(\Lambda V, d)$: minimal Sullivan algebra
$\overset{\text{def}}{\iff}$
- V: graded $\mathbb{Q}$-vector space with $V^{\leq 1} = 0$
  本当は $V^{\leq 0} = 0$ で十分だが，
  簡単のため $V^1 = 0$ も仮定する
- $\Lambda V := \mathrm{Poly}(V^{\mathrm{even}}) \otimes \mathrm{Exterior}(V^{\mathrm{odd}})$
  i.e. the free comm alg generated by V
- $(\Lambda V, d)$: DGA
- $d(V) \subset \Lambda^{\geq 2} V$ ← minimality

定義 7.3.

quasi-isom

定理 7.4.

$A^*_{\mathrm{PL}}(X) \simeq^q C^*(X)$

定義 7.5.

Sullivan model

例 7.6.

sphere

7.2 Formality

定義 7.7.

formality

定理 7.8 ([DGMS75])**.**

cpt Kaehler manifold is formal

例 7.9.

- Sphere
- wedge
- connected sum

7.3 Rational homotopy groups

定理 7.10.

$V \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_*, \mathbb{Q})$

References

[Bau81] Hans Joachim Baues. *Commutator calculus and groups of homotopy classes*. English. Vol. 50. Lond. Math. Soc. Lect. Note Ser. Cambridge University Press, Cambridge. London Mathematical Society, London, 1981. Zbl: 0473.55001 (cit. on p. 12).

[BM51] A. L. Blakers and W. S. Massey. "The homotopy groups of a triad I". English. In: *Ann. Math. (2)* 53 (1951), pp. 161–205. ISSN: 0003-486X. DOI: 10.2307/1969346. Zbl: 0042.17301 (cit. on p. 6).

[CS52a] Henri Cartan and Jean-Pierre Serre. "Espaces fibrés et groupes d'homotopie. I. Constructions générales". French. In: *C. R. Acad. Sci., Paris* 234 (1952), pp. 288–290. ISSN: 0001-4036. Zbl: 0048.41303 (cit. on p. 18).

[CS52b] Henri Cartan and Jean-Pierre Serre. "Espaces fibrés et groupes d'homotopie. II. Applications". French. In: *C. R. Acad. Sci., Paris* 234 (1952), pp. 393–395. ISSN: 0001-4036. Zbl: 0049.40101 (cit. on p. 18).

[CMN79] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer. "The double suspension and exponents of the homotopy groups of spheres". English. In: *Ann. Math. (2)* 110 (1979), pp. 549–565. ISSN: 0003-486X. DOI: 10.2307/1971238. Zbl: 0443.55009 (cit. on p. 11).

[DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. "Real homotopy theory of Kähler manifolds". English. In: *Invent. Math.* 29 (1975), pp. 245–274. ISSN: 0020-9910. DOI: 10.1007/BF01389853. Zbl: 0312.55011 (cit. on p. 21).

[Die08] Tammo tom Dieck. *Algebraic topology*. English. EMS Textb. Math. Zürich: European Mathematical Society (EMS), 2008. ISBN: 978-3-03719-048-7. DOI: 10.4171/048. Zbl: 1156.55001 (cit. on p. 14).

[EH62] Beno Eckmann and Peter J. Hilton. "Group-like structures in general categories. I. Multiplications and comultiplications". English. In: *Math. Ann.* 145 (1962), pp. 227–255. ISSN: 0025-5831. DOI: 10.1007/BF01451367. URL: https://eudml.org/doc/160908. Zbl: 0099.02101 (cit. on p. 4).

[Fre37] Hans Freudenthal. "Über die Klassen der Sphärenabbildungen. I. Große Dimensionen". German. In: *Compos. Math.* 5 (1937), pp. 299–314. ISSN: 0010-437X. URL: https://eudml.org/doc/88683. Zbl: 0018.17705 (cit. on p. 6).

[Gra69] Brayton Gray. "On the sphere of origin of infinite families in the homotopy groups of spheres". English. In: *Topology* 8 (1969), pp. 219–232. ISSN: 0040-9383. DOI: 10.1016/0040-9383(69)90012-3. Zbl: 0176.52502 (cit. on p. 12).

[Gys41] Werner Gysin. "Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten". German. In: *Comment. Math. Helv.* 14 (1941), pp. 61–122. ISSN: 0010-2571. DOI: 10.1007/BF02565612. URL: https://eudml.org/doc/138779. Zbl: 0026.27003 (cit. on p. 15).

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0. URL: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html. MR: 1867354 (cit. on pp. 1, 5–7, 13, 14, 16, 18, 20).

[Hat] Allen Hatcher. *Spectral Sequences*. URL: https://pi.math.cornell.edu/~hatcher/AT/SSpage.html (cit. on pp. 1, 10, 20).

[Hur35] W. Hurewicz. "Beiträge zur Topologie der Deformationen. II: Homotopie- und Homologiegruppen". German. In: *Proc. Akad. Wet. Amsterdam* 38 (1935), pp. 521–528. ISSN: 0370-0348. Zbl: 0011.37101 (cit. on p. 5).

[Nei10] Joseph Neisendorfer. *Algebraic methods in unstable homotopy theory*. English. Vol. 12. Cambridge: Cambridge University Press, 2010, pp. xix + 554. ISBN: 978-0-521-76037-9/hbk. Zbl: 1190.55001 (cit. on pp. 9, 11).

[Nei81] Joseph A. Neisendorfer. "3-primary exponents". English. In: *Math. Proc. Camb. Philos. Soc.* 90 (1981), pp. 63–83. ISSN: 0305-0041. DOI: 10.1017/S0305004100058539. Zbl: 0483.55007 (cit. on p. 11).

[Ser51] Jean-Pierre Serre. "Homologie singulière des espaces fibrés. Applications". In: *Ann. of Math. (2)* 54 (1951), pp. 425–505. ISSN: 0003-486X. DOI: 10.2307/1969485. MR: 0045386 (cit. on pp. 10, 14, 15).

[Ser53] Jean-Pierre Serre. "Groupes d'homotopie et classes de groupes abéliens". In: *Ann. of Math. (2)* 58 (1953), pp. 258–294. ISSN: 0003-486X. DOI: 10.2307/1969789. MR: 0059548 (cit. on pp. 9–11, 20).

[Tod56] Hirosi Toda. *On the double suspension E^2*. English. J. Inst. Polytechn., Osaka City Univ., Ser. A 7, 103-145 (1956). 1956. Zbl: 0073.18303 (cit. on p. 11).

[Wan49] Hsien-Chung Wang. "The homology groups of the fibre bundles over a sphere". English. In: *Duke Math. J.* 16 (1949), pp. 33–38. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-49-01603-8. Zbl: 0033.30801 (cit. on p. 14).

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. English. Vol. 38. Cambridge: Cambridge University Press, 1994, pp. xiv + 450. ISBN: 0-521-43500-5/hbk. Zbl: 0797.18001 (cit. on p. 17).

[Wika] Wikipedia. *Clutching construction*. URL: https://en.wikipedia.org/wiki/Clutching_construction (cit. on p. 14).

[Wikb] Wikipedia. *Eckmann–Hilton duality*. URL: https://en.wikipedia.org/wiki/Eckmann%E2%80%93Hilton_duality (cit. on p. 17).

[Wikc] Wikipedia. *Homotopy groups of spheres*. URL: https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres (cit. on p. 2).

[Wikd] Wikipedia. *Hopf fibration*. URL: https://en.wikipedia.org/wiki/Hopf_fibration (cit. on p. 13).

[河澄 22] 河澄 響矢. **トポロジーの基礎**. Vol. 下. 東京大学出版会, 2022. URL: https://cir.nii.ac.jp/crid/1971149384837313033 (cit. on p. 6).

[小中菅 67] 小松 醇郎, 中岡 稔, and 菅原 正博. **位相幾何学 *I***. 現代數學. 岩波書店, 1967. URL: https://cir.nii.ac.jp/crid/1970586434816528293. NDL PID: 2422209. [NDLで本文を閲覧] (cit. on p. 14).

[玉木 20] 玉木 大. **ファイバー束とホモトピー**. 森北出版, 2020. URL: https://cir.nii.ac.jp/crid/1970023484843984537 (cit. on pp. 13, 17).

[服部 91] 服部 晶夫. **位相幾何学**. 岩波基礎数学選書. 岩波書店, 1991. URL: https://cir.nii.ac.jp/crid/1970867909779844909 (cit. on pp. 1, 14).

[NISS] **多元入門シリーズ**. URL: https://sites.google.com/view/nagoya-intro-seminar-series/ (cit. on p. 1).

国立国会図書館の本文リンクについて：「NDL で本文を閲覧」と表示された資料の一部は，個人向けデジタル化資料送信サービスの対象です．自宅等からの閲覧には，日本国内在住の登録利用者（本登録）であることと，利用規約への同意が必要です．本登録は無料ですが，氏名・生年月日・現住所を確認できる本人確認書類が必要です．利用者登録の詳細